15th HANOI OPEN MATHEMATICS COMPETITION
Hanoi, Vietnam 2018

INDIVIDUAL CONTEST - SENIOR SECTION

Time limit: 120 minutes

Instructions:

- Write down your name, your ID number and your team's name on the first page.
- Answer all 15 questions. In Section A, each question is worth 5 points. In Section B, each question is worth 10 points. In Section C, each question is worth 15 points. The total is 150 points. There is no penalty for a wrong answer.
- For Section A (questions 1-5), circle the correct answer A, B, C, D or E. For Section B (questions 6-10), fill your answer in the space provided at the end of each question. For Section C (questions 11-15), write your detailed solution in the space provided at the end of each question.
- Diagrams shown may not be drawn to scale.
- No calculators, protractors or electronic devices are allowed to use.
- Answers must be in pencil, blue or black ball-point pen.
- All papers shall be collected at the end of the test.

Name: ....
Team: ....
ID number: ....

For jury only

15th HANOI OPEN MATHEMATICS COMPETITION
Hanoi, Vietnam 2018

Individual Contest - Senior Section

For jury only

For Jury Only									
Question	Section A					Section B Subtotal	Section C Subtotal	Total	Signature
	1	2	3	4	5				
Score									

SECTION A. *Circle the correct answer* A, B, C, D *or* E.

Question 1. How many rectangles can be formed by the vertices of a cube? *(Note: square is also a special rectangle)*.

A. 6 **B.** 8 **C.** 12 **D.** 18 **E.** 16

Question 2. What is the largest area of a regular hexagon that can be drawn inside the equilateral triangle of side 3?

A. $3\sqrt{7}$ **B.** $\frac{3\sqrt{3}}{2}$ **C.** $2\sqrt{5}$ **D.** $\frac{3\sqrt{3}}{8}$ **E.** $\frac{3\sqrt{5}}{2}$

Question 3. How many integers n are there those satisfy the following inequality

$$n^4 - n^3 - 3n^2 - 3n - 17 < 0?$$

A. 4 **B.** 6 **C.** 8 **D.** 10 **E.** 12

Question 4. Let

$$a = (\sqrt{2}+\sqrt{3}+\sqrt{6})(\sqrt{2}+\sqrt{3}-\sqrt{6})(\sqrt{3}+\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2}-\sqrt{3})$$
$$b = (\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}-\sqrt{3}).$$

The difference $a - b$ belongs to the set:

A. $(-\infty, -4)$ **B.** $[-4, 0)$ **C.** $\{0\}$ **D.** $(0, 4]$ **E.** $(4, \infty)$

Question 5. The center of a circle and nine randomly selected points on this circle are colored in red. Every pair of those points is connected by a line segment, and every point of intersection of two line segments inside the circle is colored in red. What is the largest possible number of red points?

A. 235 **B.** 245 **C.** 250 **D.** 220 **E.** 265

For jury only

For Jury Only						
Question	Section B					Total
	6	7	8	9	10	
Score						

SECTION B. *Fill your answer in the space provided at the end of the question.*

Question 6. Write down all real numbers (x, y) satisfying two conditions:

$$x^{2018} + y^2 = 2, \text{ and } x^2 + y^{2018} = 2.$$

Answer. ....

Question 7. Let $\{u_n\}_{n\geq 1}$ be given sequence satisfying the conditions:

$$u_1 = 0, u_2 = 1, u_{n+1} = u_{n-1} + 2n - 1 \quad \text{for} \quad n \geq 2.$$

1) Calculate u_5.

2) Calculate $u_{100} + u_{101}$.

Answer. 1) .... 2)...............................

Question 8. Let P be a point inside the square $ABCD$ such that $\angle PAC = \angle PCD = 17^o$ (see Figure 1). Calculate $\angle APB$?

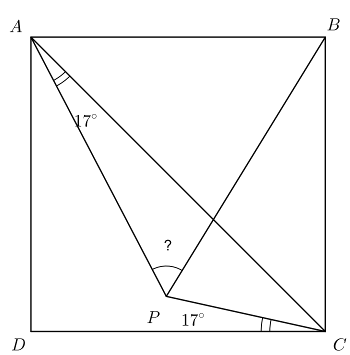

Figure 1

Answer. ....

For jury only

SECTION B. *Fill your answer in the space provided at the end of the question.*

Question 9. How many ways of choosing four edges in a cube such that any two among those four choosen edges have no common point.

Answer. ....

Question 10. There are 100 school students from two clubs A and B standing in circle. Among them 62 students stand next to at least one student from club A, and 54 students stand next to at least one student from club B.

1) How many students stand side-by-side with one friend from club A and one friend from club B?

2) What is the number of students from club A?

Answer. 1) ....; 2) ....

For jury only

For Jury Only						
Question	Section C					Total
	11	12	13	14	15	
Score						

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 11. Find all positive integers k such that there exists a positive integer n, for which $2^n + 11$ is divisible by $2^k - 1$.

Solution:

For jury only

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 12. Let ABC be an acute triangle with $AB < AC$, and let BE and CF be the altitudes. Let the median AM intersect BE at point P, and let line CP intersect AB at point D (see Figure 2). Prove that $DE \parallel BC$, and AC is tangent to the circumcircle of $\triangle DEF$.

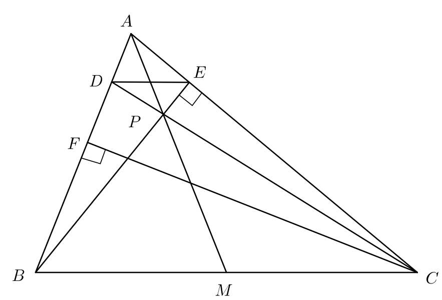

Figure 2

Solution:

For jury only

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 13. For a positive integer n, let $S(n), P(n)$ denote the sum and the product of all the digits of n respectively.

1) Find all values of n such that $n = P(n)$.

2) Determine all values of n such that $n = S(n) + P(n)$.

Solution:

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 14. Let a, b, c denote the real numbers such that $1 \leq a, b, c \leq 2$. Consider

$$T = (a-b)^{2018} + (b-c)^{2018} + (c-a)^{2018}.$$

Determine the largest possible value of T.

Solution:

For jury only

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 15. There are n distinct straight lines on a plane such that every line intersects exactly 12 others. Determine all the possible values of n.

Solution:

15th HANOI OPEN MATHEMATICS COMPETITION
Hanoi, Vietnam 2018

Individual Contest - Senior Section

For Jury Only

Question	Section A					Section B	Section C	Total	Signature
	1	2	3	4	5				
Score									

SECTION A. *Circle the correct answer* A, B, C, D *or* E.

Question 1. How many rectangles can be formed by the vertices of a cube? *(Note: square is also a special rectangle)*.

A. 6 **B.** 8 **C.** 12 **D.** 18 **E.** 16

Solution. Correct Answer **C**. There are 6 squares on 6 faces of the cube. There are 4 diagonals of the cube that have the same length and pass through the center of the cube. Every two diagonals intersect at their midpoint, hence they form a rectangle. So, there are $6+\binom{4}{2}=12$ rectangles. □

Question 2. What is the largest area of a regular hexagon that can be drawn inside the equilateral triangle of side 3?

A. $3\sqrt{7}$ **B.** $\frac{3\sqrt{3}}{2}$ **C.** $2\sqrt{5}$ **D.** $\frac{3\sqrt{3}}{8}$ **E.** $\frac{3\sqrt{5}}{2}$

Solution. Correct Answer **B**. Suppose that the regular hexagon H with side a is inside the triangle equilateral triangle with side 3. Then, the inscribed circle of H is also inside the triangle, and its radius is equal to $a\sqrt{3}/2$. On the other hand, the largest circle in the given equilateral triangle is its inscribed circle whose radius is $\sqrt{3}/2$. It follows that $a \le 1$ and the answer is $6\sqrt{3}/4 = 3\sqrt{3}/2$. □

Question 3. How many integers n are there those satisfy the following inequality

$$n^4 - n^3 - 3n^2 - 3n - 17 < 0?$$

A. 4 **B.** 6 **C.** 8 **D.** 10 **E.** 12

Solution. Correct Answer A. We have

$$(n+1)^3 + 16 > n^4 \geq 0,$$

which implies $n \geq -3$. For $n \geq 4$ we have

$$n^4 - (n+1)^3 \geq 3n^3 - 3n^2 - 3n - 1 \geq 12n^2 - 3n^2 - 3n - 1 = n(n-3) + 8n^2 - 1 > 16.$$

Hence, $-3 \leq n \leq 3$. By directly calculation we obtain $n = -1, 0, 1, 2$. □

Question 4. Let

$$a = (\sqrt{2}+\sqrt{3}+\sqrt{6})(\sqrt{2}+\sqrt{3}-\sqrt{6})(\sqrt{3}+\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2}-\sqrt{3})$$
$$b = (\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}-\sqrt{3}).$$

The difference $a - b$ belongs to the set:

A. $(-\infty, -4)$ **B.** $[-4, 0)$ **C.** $\{0\}$ **D.** $(0, 4]$ **E.** $(4, \infty)$

Solution. Correct Answer B. Note that

$$(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = 2(x^2y^2 + y^2z^2 + z^2x^2) - x^4 - y^4 - z^4.$$

We get $a - b = 2(2+3)(6-5) - 6^2 + 5^2 = -1$.

In fact, we can also calculate a, b as follows: for $(x, y, z) = (\sqrt{2}, \sqrt{3}, \sqrt{6})$ we get $a = 23$, and for $(x, y, z) = (\sqrt{2}, \sqrt{3}, \sqrt{5})$ we get $b = 24$. It follows that $a - b = -1$. □

Question 5. The center of a circle and nine randomly selected points on this circle are colored in red. Every pair of those points is connected by a line segment, and every point of intersection of two line segments inside the circle is colored in red. What is the largest possible number of red points?

A. 235 **B.** 245 **C.** 250 **D.** 220 **E.** 265

Solution. Correct Answer D. Remark that a convex quadrilateral has exactly one intersection which is the intersection of its two diagonals. Consider 9 points on the circle, which give at most $\binom{9}{4} = 126$ intersections. Considering the center and three points on the circle, there are at most $\binom{9}{3} = 84$ intersections. So there are at most 126 + 84+10 = 220 red points. □

15$^{\text{th}}$ HANOI OPEN MATHEMATICS COMPETITION
Individual Contest - Senior Section

ANSWER AND SOLUTION

Question	Section B					Total
	6	7	8	9	10	
Score						

SECTION B. *Fill your answer in the space provided at the end of the question.*

Question 6. Write down all real numbers (x, y) satisfying two conditions:

$$x^{2018} + y^2 = 2, \text{ and } x^2 + y^{2018} = 2.$$

Answer. ....

Solution. $(1, 1), (1, -1), (-1, 1), (-1, -1)$.

For jury: The maximal score for uncompleted answer is 8 points (2 points for each one of those).

If $x^2 > 1$ then $x^{2018} > x^2 > 1 \implies y^2 < 1 \implies y^2 > y^{2018}$. Thus $x^{2018} + y^2 > x^2 + y^{2018}$ (contradiction). Analogically, if $x^2 < 1 \implies x^{2018} + y^2 < x^2 + y^{2018}$ (contradiction). Therefore $x^2 = y^2 = 1$.

Question 7. Let $\{u_n\}_{n \geq 1}$ be given sequence satisfying the conditions:

$$u_1 = 0, u_2 = 1, u_{n+1} = u_{n-1} + 2n - 1 \quad \text{for} \quad n \geq 2.$$

1) Calculate u_5.

2) Calculate $u_{100} + u_{101}$.

Answer. 1) .... 2)...............................

Solution. 1)...10 **(4 points)**; 2) ...10000 **(6 points).**

1) It is easy to see that $u_2 = 1, u_3 = 3, u_4 = 6, u_5 = 10$.

2) We can prove by induction that

$$u_n = \frac{n(n-1)}{2} \text{ for every } n \geq 1.$$

Therefore,

$$u_n + u_{n+1} = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2 \text{ for every } n \geq 1.$$

Thus, $u_{100} + u_{101} = 10000$.

Question 8. Let P be a point inside the square $ABCD$ such that $\angle PAC = \angle PCD = 17^o$ (see Figure 1). Calculate $\angle APB$?

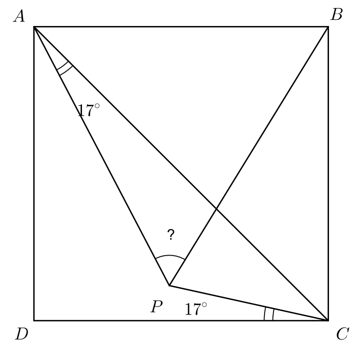

Figure 1

Answer. ....

Solution. ......62^o.

Let $\mathcal{O}(B, BA)$ denote the circle with its center B and radius BA. Let Q denote the point of intersection of ray CP with $\mathcal{O}(B, BA)$. We have $\angle DCQ = 17^o$, which implies $\angle CAQ = \angle CAP = 17^o$. Hence, $Q \equiv P$. We have

$$\angle ABP = 2\angle ACP = 2(45^o - 17^o) = 56^o;$$
$$\angle APB = 180^o - (56^o + 45^o + 17^o) = 62^o.$$

The answer is 62^o.

Question 9. How many ways of choosing four edges in a cube such that any two among those four choosen edges have no common point.

Answer. ....

Solution. ...9

Results: There are 9 choices. Consider 2 cases: all edges are parallel (3 choices) or two pairs of parallel edges (6 choices).

Question 10. There are 100 school students from two clubs A and B standing in circle. Among them 62 students stand next to at least one student from club A, and 54 students stand next to at least one student from club B.

1) How many students stand side-by-side with one friend from club A and one friend from club B?

2) What is the number of students from club A?

Answer. 1) ....; 2) ....

Solution. 1)....16 **(4 points)** 2) .....54 **(6 points)**.

Let x denote the number of students standing between 2 students from club A, i.e. $(A - * - A)$; y denote the number of students standing between 2 students from club B, i.e.

$(B - * - B)$; and z denote the number of students standing between 1 student from A and 1 student from B, i.e. $(A - * - B)$ or $(B - * - A)$.

Thus we have $x + y + z = 100$; $x + z = 62$; $y + z = 54$. From those equations we have $z = 62 + 54 - 100 = 16$, $x = 62 - 16 = 46$ and $y = 54 - 16 = 38$.

Moreover each student from club A has exactly 2 neighbours. Therefore $2x + z = 2|A|$, where $|A|$ is the number of students from club A. Thus $|A| = (2 * 46 + 16)/2 = 54$.

15^{th} HANOI OPEN MATHEMATICS COMPETITION
Individual Contest - Senior Section

ANSWER AND SOLUTION

Question	Section C					Total
	11	12	13	14	15	
Score						

SECTION C. *Write your detailed solution in the space provided at the end of the question.*

Question 11. Find all positive integers k such that there exists a positive integer n, for which $2^n + 11$ is divisible by $2^k - 1$.

Solution. Suppose $n \in \mathbb{Z}^+$ is a positive integer such that $2^n + 11$ is divisible by $2^k - 1$. Let $n = qk + r$ where q, r are nonnegative integers and $0 \le r < k$.

We have:

$$2^n + 11 = 2^{kq+r} + 11 = 2^r(2^{kq} - 1) + (2^r + 11) \vdots 2^k - 1 \Rightarrow 2^r + 11 \vdots 2^k - 1. \quad (\mathbf{5\ points})$$

Thus we have $2^r + 11 \ge 2^k - 1$ and $r < k$. Therefore $k \le 4$ **(5 points)**. By checking for $k = 1, 2, 3, 4$, we obtain $k = 1, 2, 4$ **(5 points)**

Question 12. Let ABC be an acute triangle with $AB < AC$, and let BE and CF be the altitudes. Let the median AM intersect BE at point P, and let line CP intersect AB at point D (see Figure 2). Prove that $DE \parallel BC$, and AC is tangent to the circumcircle of $\triangle DEF$.

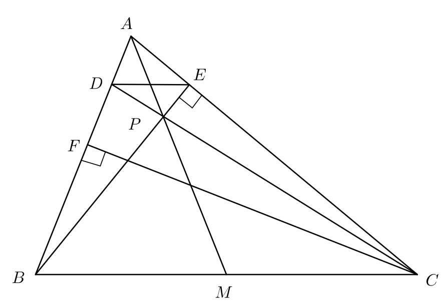

Figure 2

Solution. 1) Let $ED' \parallel BC$ (with $D' \in AB$). We see that P is the common point of BE, CD', AM in the trapezoid $BCED'$. It follows $D' \equiv D, DE \parallel BC$ **(5 points)**.

2) Let H be the point of intersection of BE and CF and let K denote the point of intersection of AH and DE. We see that the quadrilateral $HKDF$ is inscribed in a circle, hence $AD.AF = AK.AH$. **(5 points)**! In other hand, we have $AE^2 = AK.AH$. By the above equalities we deduce $AE^2 = AD.AF$, as desired. **(5 points)**.

Question 13. For a positive integer n, let $S(n), P(n)$ denote the sum and the product of all the digits of n respectively.

1) Find all values of n such that $n = P(n)$.

2) Determine all values of n such that $n = S(n) + P(n)$.

Solution. Let $n = \overline{a_k a_{k-1} \dots a_1 a_0} = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + 10a_1 + a_0$.

1) If $k \geq 1$, then

$$\begin{aligned} n - P(n) &= a_k 10^k + a_{k-1} 10^{k-1} + \cdots + 10a_1 + a_0 - (a_k a_{k-1} \dots a_0) \\ &\geq a_k (10^k - 9^k) > 0. \end{aligned}$$

Thus $n = P(n) \Leftrightarrow k = 0 \implies n = 1, 2, 3, 4, 5, 6, 7, 8, 9.$ **(5 points)**.

2) Notice that in this case n has at least 2 digits. Moreover we have

$$\begin{aligned} &n - S(n) - P(n) \\ &= a_k 10^k + a_{k-1} 10^{k-1} + \cdots + 10a_1 + a_0 - (a_k + \cdots + a_0) - a_k a_{k-1} \dots a_1 a_0 \\ &= a_k (10^k - a_{k-1} \dots a_1 a_0 - 1) + a_{k-1}(10^{k-1} - 1) + \cdots + a_1.9 \\ &\geq a_k (10^k - 9^k - 1) \geq 0. \end{aligned}$$

The equality occurs if $k = 1$. **(5 points)**. We find $n = 10a_1 + a_0 = a_1 + a_0 + a_1 a_0$ i.e. $10 = 1 + a_0$ and $a_0 = 9$. All numbers $19, 29, 39, 49, 59, 69, 79, 89, 99$ are satisfying the condition. **(5 points)**.

Answer: $19, 29, 39, 49, 59, 69, 79, 89, 99$.

Question 14. Let a, b, c denote the real numbers such that $1 \leq a, b, c \leq 2$. Consider

$$T = (a-b)^{2018} + (b-c)^{2018} + (c-a)^{2018}.$$

Determine the largest possible value of T.

Solution. Without loss of generality, one can assume that

$$1 \leq c \leq b \leq a \leq 2. \tag{1}$$

Hence, $0 \leq a - b \leq 1$ and $(a-b)^{2018} \leq a - b$, and the equality holds if $a = b$ or $(a, b) = (2, 1)$. **(5 points)**. Similarly, $0 \leq b - c \leq 1$ then $(b-c)^{2018} \leq b - c$, $0 \leq a - c \leq 1$ then $(c-a)^{2018} \leq a - c$. **(5 points)**. Hence,

$$T = (a-b)^{2018} + (b-c)^{2018} + (c-a)^{2018} \leq a - b + b - c + a - c = 2(a-c) \leq 2.$$

The equality holds if $(a, b, c) = (2, 2, 1)$ or $(a, b, c) = (2, 1, 1)$. So, $\max T = 2$. **(5 points)**.

Question 15. There are n distinct straight lines on a plane such that every line intersects exactly 12 others. Determine all the possible values of n.

Solution. Suppose that we have k seperate groups of parallel straight lines. Every straight line of each group will intersects all other straight lines of other $(k-1)$ groups. Hence, 12 is the number of straight lines in every $(k-1)$ groups. It follows all k groups consist the same number, that called q. **(5 points)**. We then have $12 = q(k-1)$ and $n = kq$. This follows $12 + q = kq$. **(5 points)**. We deduce 12 is divisible by q, i.e. $q = 1, 2, 3, 4, 6, 12$. As above, the identity $n = 12 + q$ gives $n \in \{13, 14, 15, 16, 18, 24\}$. **(5 points)**.