15^{th} HANOI OPEN MATHEMATICS COMPETITION

Hanoi, Vietnam 2018

INDIVIDUAL CONTEST – JUNIOR SECTION

ANSWER KEY

Time limit: 120 minutes

Instructions:

- Write down your name, your contestant number and your team's name on the first page.
- Answer all 15 questions. In Section A, each question is worth 5 points. In Section B, each question is worth 10 points. In Section C, each question is worth 15 points. The total is 150 points. There is no penalty for a wrong answer.
- For questions 1-5, circle the correct answer A, B, C, D or E. For questions 6-10, fill your answer in the space provided at the end of each question. For questions 11-15, write your detailed solution in the space provided at the end of each question.
- Diagrams shown may not be drawn to scale.
- No calculators, protractors or electronic devices are allowed to use.
- Answers must be in pencil, blue or black ball-point pen.
- All papers shall be collected at the end of the test.

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Section A. There are 5 questions. Select and circle the right solution.

Q1. Let x and y be real numbers satisfying the conditions $x + y = 4$ and $xy = 3$. Compute the value of $(x - y)^2$.

A. 0 **B.** 1 **C.** 4 **D.** 9 **E.** None of above

Solution. The answer is $\boxed{C}$

We have $(x - y)^2 = (x + y)^2 - 4xy = 16 - 12 = \boxed{4}$

Q2. Let $f(x)$ be a polynomial such that

$$2f(x) + f(2 - x) = 5 + x$$

for any real number x. Find the value of $f(0) + f(2)$.

A. 4 **B.** 0 **C.** 2 **D.** 3 **E.** 1

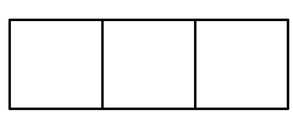
Solution. The answer is $\boxed{A}$

We have $x = 0 \Rightarrow 2f(0) + f(2) = 5$ (1)

$x = 2 \Rightarrow 2f(2) + f(0) = 7$ (2)

Then $(1)+(2)$ gives: $f(0) + f(2) = \dfrac{5+7}{3} = \boxed{4}$.

Q3. There are 3 unit squares in a row as shown in the figure below. Each side of this figure is painted by one of three colors: Blue, Green or Red. It is known that for any square, all the three colors are used and no two adjacent sides have the same color. Find the number of possible colorings.



A. 48 **B.** 96 **C.** 108 **D.** 192 **E.** 216

Solution. The answer is $\boxed{D}$

There are 2 ways to choose the 2 sides which have the same color in the first square. So the first square has $2\cdot 3! = 12$ colorings. For the second square: the color of the left side is fixed, if the left side and the right side have the same color then there are $2! = 2$ colorings for the remaining sides; if the upper side and the lower side have the same color, one also has $2! = 2$ colorings for the remaining sides. So the second square has 4 colorings. The argument is similar for the next square. Answer: $12\times 4\times 4 = \boxed{192}$.

Q4. Find the number of distinct real roots of the following equation.

$$x^2 + \frac{9x^2}{(x+3)^2} = 40$$

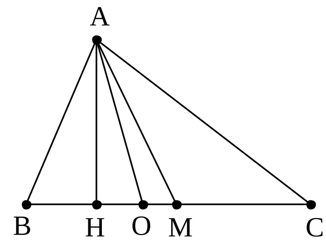
A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

Solution. The answer is $\boxed{C}$

$$\begin{aligned}
x^2 + \frac{9x^2}{(x+3)^2} = 40 &\Leftrightarrow \left(x - \frac{3x}{x+3}\right)^2 + 6\frac{x^2}{x+3} = 40\\
&\Leftrightarrow \left(\frac{x^2}{x+3}\right)^2 + 6\frac{x^2}{x+3} - 40 = 0\\
&\Rightarrow t^2 + 6t - 40 = 0 \quad \left(t = \frac{x^2}{x+3}\right)\\
&\Leftrightarrow (t-4)(t+10) = 0\\
&\Leftrightarrow \left[\begin{array}{l} t = 4 \Leftrightarrow \left[\begin{array}{l} x = 6\\ x = -2\end{array}\right.\\ t = -10 \Rightarrow \text{no real root}\end{array}\right.
\end{aligned}$$

So, the equation has $\boxed{2}$ real roots.

Q5. Let ABC be an acute triangle with $AB = 3$ and $AC = 4$. Suppose that AH, AO and AM are the altitude, the bisector and the median derived from A, respectively. If $HO = 3\times OM$, then the length of BC is



A. 3 **B.** $\frac{7}{2}$ **C.** 4 **D.** $\frac{9}{2}$ **E.** 5

Solution. The answer is $\boxed{B}$.

Let $CH = x, BC = a.$ By Pythagoras' theorem,

$$\begin{cases} x^2 + AH^2 = 16 \\ (a-x)^2 + AH^2 = 9 \end{cases} \Rightarrow 2ax - a^2 = 7 \Rightarrow x = \frac{a^2+7}{2a}.$$

So $HM = CH - CM = \frac{a^2+7}{2a} - \frac{a}{2} = \frac{7}{2a}.$

By a property of the bisector AO, one has $\frac{CO}{BO} = \frac{4}{3}$, then $CO = \frac{4a}{7}$, hence

$$OM = CO - CM = \frac{4a}{7} - \frac{a}{2} = \frac{a}{14}.$$

Since $HM = 4 \times OM$, one has $\frac{7}{2a} = \frac{4a}{14} = \frac{2a}{7}$, hence $a = \boxed{\frac{7}{2}}.$

15^{th} HANOI OPEN MATHEMATICS COMPETITION

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Section B. There are 5 questions. Fill in your answer in the space provided at the end of each question.

Q6. Nam spent 20 dollars for 20 stationery items consisting of books, pens and pencils. Each book, pen, and pencil costs 3 dollars, 1.5 dollars and 0.5 dollar, respectively. How many dollars did Nam spend for books?

Answer: ____________________

Solution. The answer is $\boxed{6}$.

Let *a*, *b*, *c* be the number of books, pens and pencils respectively.

Then $a+b+c=20$ & $3a+1.5b+0.5c=20.$

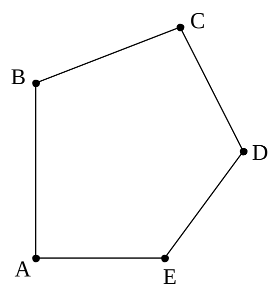
$=> 5a+2b=20 \Rightarrow b \vdots 5; 0<b<10 \Rightarrow a=2; b=5; c=13.$

$\Rightarrow$ The answer is $2\times 3=\boxed{6}$.

Q7. Suppose that $ABCDE$ is a convex pentagon with

$$\angle A=90°, \angle B=105°, \angle C=90°, \text{and } AB=2, BC=CD=DE=\sqrt{2}.$$

If the length of AE is $\sqrt{a}-b$ where a, b are integers, what is the value of $a+b$?

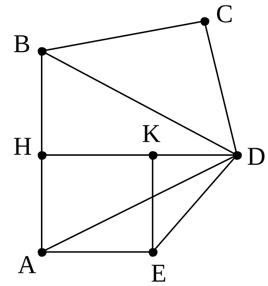


Answer: ____________________

Solution. The answer is $\boxed{4}$.

Draw $DH \perp AB, EK \perp DH$ $\left(H \in AB, K \in DH\right)$. Since BCD is a right-angled isosceles triangle, one has $BD=2.$ Also $\angle ABD=105°-45°=60°,$ so ABD is an equilateral triangle with the side length is 2, hence $DH=\sqrt{3}.$ Now $AHKE$ is a rectangle, then $EK=HA=1.$ In triangle

DKE, $\angle DKE = 90°$, $DE = \sqrt{2}$, $EK = 1$, these implies that $KD = 1$. So $AE = HD - KD = \sqrt{3} - 1$. The answer is $3 + 1 = \boxed{4}$.



Q8. Let m be a positive integer such that $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{13} = \frac{m}{13!}$. Find the remainder when m is divided by 7.

Answer: ___________________

Solution. The answer is $\boxed{1}$.

We have $m = 13! + \frac{13!}{2} + ... + \frac{13!}{12} + \frac{13!}{13}$

$$\Rightarrow m \equiv \frac{13!}{7} \equiv 1.2.3...6.8...13 \equiv (1.2.3...6)^2 \pmod 7$$
$$\equiv 720^2 \pmod 7$$
$$\equiv (-1)^2 \equiv 1 \pmod 7$$

The answer is $\boxed{1}$.

Q9. There are three polygons and the area of each one is 3. They are drawn inside a square of area 6. Find the greatest value of a such that among those three polygons, we can always find two polygons so that the area of their overlap is not less than a.

Answer: ___________________

Solution. The answer is $\boxed{1}$.

Denote by S_i the area of the polygon i with $i = \overline{1,3}$;

S_{ij} the area of the overlap of polygons i and j with $1 \le i < j \le 3$;

S_{123} the area of the overlap of these 3 polygons;

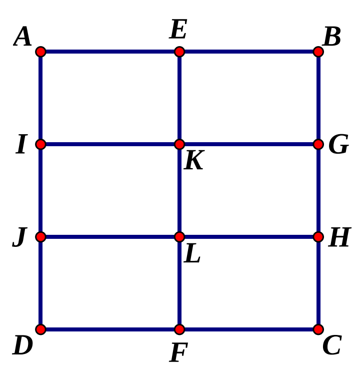
S the covering area of these 3 polygons.

Then

$$\begin{aligned} 6 \ge S &= S_1 + S_2 + S_3 - S_{12} - S_{13} - S_{23} + S_{123} \\ &= 9 + S_{123} - (S_{12} + S_{13} + S_{23}) \\ &\ge 9 - (S_{12} + S_{13} + S_{23}) \\ \Rightarrow S_{12} &+ S_{23} + S_{13} \ge 3. \end{aligned}$$

$\Rightarrow$ there exists $S_{ij} \ge 1$. Otherwise, we have the following example, where

$$S_{ABGKLJ} = S_{IKLHCD} = S_{EBCF} = 3 \text{ and } S_{12} = S_{13} = S_{23} = 1.$$



$\Rightarrow$ The greatest value of a is $\boxed{1}$.

Q10. Let $T = \frac{1}{4}x^2 - \frac{1}{5}y^2 + \frac{1}{6}z^2$ where x, y, z are real numbers such that $1 \le x,\ y,\ z \le 4$ and $x - y + z = 4$.

Find the smallest value of $10 \times T$.

Answer: ____________________

Solution. The answer is $\boxed{23}$.

We have $(x-2)^2 \ge 0 \Rightarrow x^2 \ge 4x - 4$

$$(y-1)(4-y)\geq 0 \Rightarrow -y^2 \geq -5y+4$$

$$(z-3)^2 \geq 0 \Rightarrow z^2 \geq 6z-9$$

$$\Rightarrow T=\frac{1}{4}x^2-\frac{1}{5}y^2+\frac{1}{6}z^2 \geq \frac{1}{4}\left(4x-4\right)+\frac{1}{5}(-5y+4)+\frac{1}{6}(6z-9)=x-y+z-\frac{17}{10}=\frac{23}{10}$$

$$\Rightarrow 10\times T \geq 23$$

$$\Rightarrow \min(10\times T)=23 \text{ when } x=2,\ y=1,\ z=3.$$

Section C. Answer the following 5 questions. Present your detailed solution in the space provided.

Q11. Find all pairs of nonnegative integers $(x;\ y)$ for which $(xy+2)^2 = x^2 + y^2$.

Solution.

$$(xy+2)^2 = x^2 + y^2 \Leftrightarrow (xy+3)^2 - 5 = (x+y)^2 \Leftrightarrow (xy+3)^2 - (x+y)^2 = 5$$

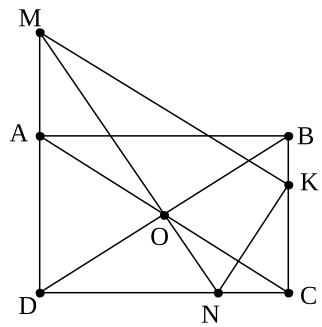
$$\Leftrightarrow (xy+x+y+3)(xy-x-y+3) = 5.$$ **(5 *points*)**

Since x and y are nonnegative then

$$xy+x+y+3 \geq 3 \Rightarrow \begin{cases} xy+x+y+3=5 \\ xy-x-y+3=1 \end{cases} \Rightarrow \begin{cases} xy=0 \\ x+y=2 \end{cases}$$ **(5 *points*)**

$$\Rightarrow (x, y) \in \{(0,\ 2);(2,\ 0)\}.$$ **(5 *points*)**

Q12. Let $ABCD$ be a rectangle with $45^0 < ADB < 60^0$. The diagonals AC and BD intersect at O. A line passing through O and perpendicular to BD meets AD and CD at M, N respectively. Let K be a point on side BC such that $MK \| AC$. Show that $\angle MKN = 90°$.



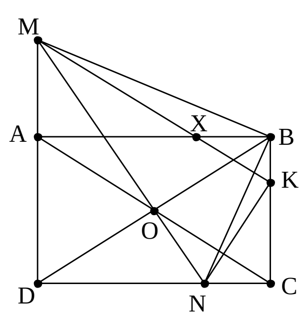
Solution 1. Let $AB \cap MK = \{X\}$. By symmetry, $\angle MBN = \angle MDN = 90°$. **(*2 points*)**

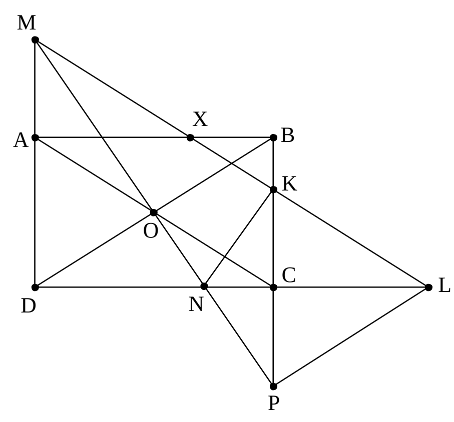
So $\angle MBA = \angle NBC$ then $\Delta BAM \sim \Delta BCN$. **(*3 points*)**

Therefore $\frac{CN}{AM} = \frac{BC}{BA} = \frac{KC}{XA}$ (since $XK \| AC$). Hence $\Delta KCN \sim \Delta XAM \sim \Delta XBK$. **(*5 points*)**

This implies that $\angle XKB = \angle KNC$. But $\angle KNC + \angle CKN = 90°$

then $\angle XKB + \angle CKN = 90°$ thus $\angle XKN = 90°$. **(*5 points*)**





Solution 2. Let $MK \cap CD = \{L\}, MK \cap \{AB\} = X, MN \cap BC = \{P\}$.

One has $AMKC, AMCP, AXLC$ are parallelograms, hence $CK = AM = CP, AX = CL$. **(*5 points*)**

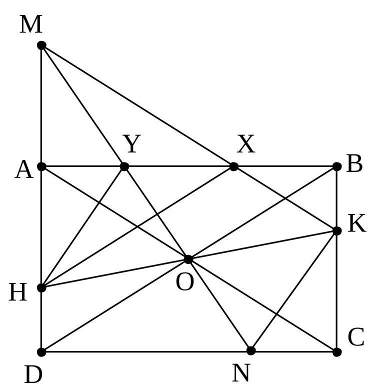
By Thales' theorem, $\frac{CL}{CP}=\frac{AX}{CK}=\frac{AB}{BC}=\frac{CD}{BC}$. Therefore $LP \parallel BD$, but $MN \perp BD$ then $\angle NPL = 90°$. **(5 *points*)**

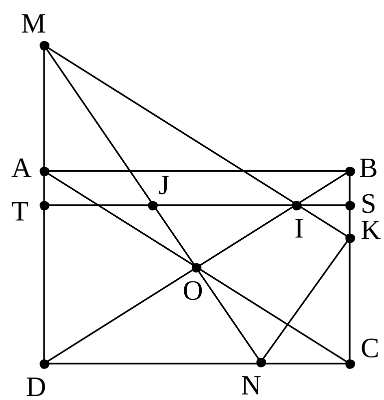
Since $CK = CP$, the points K and P are symmetric about NL. This implies that $\angle NKP = \angle NPL = 90°$ or $\angle MKN = 90°$. **(5 *points*)**

Solution 3. Let $MK \cap AB = \{X\}, OK \cap AD = \{H\}, MO \cap AB = \{Y\}$. One has $HYKN, DHBK$ are parallelograms, hence $HY \parallel NK, BK = DH$. **(5 *points*)**

Since $MK \parallel AC$, $\frac{XB}{AB}=\frac{BK}{BC}=\frac{DH}{AD}$. Then $HX \parallel BD$, but $MN \perp BD$ then $MN \perp HX$. **(5 *points*)**

In triangle MHX, $XA \perp HM$, $AO \perp HX$ so Y is the orthocenter. This implies that $HY \perp MX$ and $NK \perp MK$. **(5 *points*)**





Solution 4. Let $DB \cap MK = \{I\}$. Let T, S be the midpoints of MD, BK, respectively. Since $MK \parallel AC$, one has $\angle IKB = \angle OCB = \angle IBK = \angle IDM = \angle IMD$. So $MBKD$ is an isosceles trapezoid. **(5 *points*)**

Therefore S, I, T are collinear, ST is the axes of symmetry of $MBKD$ and $TS \parallel CD$. **(5 *points*)**

Let $TS \cap MN = \{J\}$ then TJ is the mid-line of triangle MDN, hence J is the midpoint of MN. **(5 *points*)**

Note that TS and MN are the perpendicular bisectors of BK and BD, respectively. Therefore $JK = JB = JD$. Since $\angle MDN = 90°$, one has $JD = \frac{MN}{2}$, then $KJ = \frac{MN}{2}$ and it follows that $\angle MKN = 90°$. **(*5 points*)**

Solution 5. (Use inscribed quadrilateral).

It is similar to solution 1, one has $\angle MBN = \angle MDN = 90°$. **(*2 points*)**

Hence the quadrilateral $MBND$ is cyclic. **(*3 points*)**

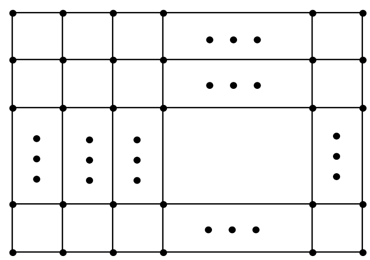
It similar to solution 4, $MBKD$ is an isosceles trapezoid, so the quadrilateral $MBKD$ is cyclic. **(*5 points*)**

Therefore M, B, K, N, D are concyclic. Then $\angle MKN = \angle MBN = 90°$. **(*5 points*)**

Q13. A competition room of HOMC has $m \times n$ students where m, n are integers larger than 2. Their seats are arranged in m rows and n columns. Before starting the test, every students take a handshake with each of his/her adjacent students (in the same row or in the same column). It is known that, there are totally 27 handshakes. Find the number of students in the room.

Solution.

Consider a $m \times n$ grid of mn points, each point is a student. Each handshake of two students can be treated as a segment joining two corresponding points. **(*5 points*)**



Hence the number of segments is $27 = (m-1)\times n + (n-1)\times m$ or $2mn - m - n = 27$. **(*5 points*)**

Therefore $4mn - 2m - 2n + 1 = 55$ then $(2m-1)(2n-1) = 55 = 5\times 11$. This implies that (m, n) is a permutation of $(3, 6)$. So the number of students is $3\cdot 6 = 18$. **(*5 points*)**

Q14. Let $P(x)$ be a polynomial with degree 2017 such that $P(k)=\frac{k}{k+1}$, $\forall k=0,1,2,...,2017.$ Calculate $P(2018)$.

Solution.

Let $Q(x)=(x+1)P(x)-x \Rightarrow \deg Q=2018$ & $Q(0)=Q(1)=Q(2)=...=Q(2017)=0.$ **(*2 points*)**

$\Rightarrow Q(x)=(x+1)P(x)-x=kx(x-1)(x-2)\times...\times(x-2017).$ **(*3 points*)**

$x=-1 \Rightarrow 1=k\times 2018! \Rightarrow k=\frac{1}{2018!}.$ **(*5 points*)**

$x=2018 \Rightarrow 2019P(2018)-2018=\frac{1}{2018!}\times 2018\times 2017\times...\times 1=1$

$\Rightarrow P(2018)=1.$ **(*5 points*)**

Q15. Find all pairs of prime numbers (p; q) such that for each pair $(p;q)$ there is a positive integer m satisfying

$$\frac{pq}{p+q}=\frac{m^2+6}{m+1}.$$

Solution. $(p;q)=\boxed{(7;7)}$.

$$\frac{pq}{p+q}=\frac{(m+1)(m-1)+7}{m+1}=m-1+\frac{7}{m+1}.$$

Case 1. If $p = q$, we have $p=2(m-1)+\frac{14}{m+1}$.

So, we find a positive integer m such that p is an integer and from that to obtain p as prime number.

$\Rightarrow m=1$ and $p=q=7$. **(*5 points*)**

Case 2. If $p\neq q\Rightarrow \gcd(pq;p+q)=1$

Let $d=\gcd(m^2+6;\ m+1)\Rightarrow d\in\{1;7\}$

+) $d=1\Rightarrow \gcd(m^2+6;\ m+1)=\gcd(pq;p+q)=1\Rightarrow \begin{cases} pq=m^2+6 \\ p+q=m+1 \end{cases}$

$\Rightarrow (m+1)^2\geq 4(m^2+6)\Rightarrow m\in\varnothing$ **(*5 points*)**

+) $d=7\Rightarrow \frac{pq}{p+q}=\frac{7k^2-2k+1}{k},\ k\in\mathbb{Z}^+$

$\Rightarrow k^2\geq 4(7k^2-2k+1)\Rightarrow k\in\varnothing$ **(*5 points*)**