Hanoi Open Mathematical Competition 2016

Junior Section

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. If

$$2016 = 2^5 + 2^6 + \cdots + 2^m,$$

then m is equal to

(A): 8 (B): 9 (C): 10 (D): 11 (E): None of the above.

Anwser. (C).

Question 2. The number of all positive integers n such that

$$n + s(n) = 2016,$$

where $s(n)$ is the sum of all digits of n, is

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Anwser. (B): $n = 1989, 2007$.

Question 3. Given two positive numbers a, b such that $a^3 + b^3 = a^5 + b^5$, then the greatest value of $M = a^2 + b^2 - ab$ is

(A): $\frac{1}{4}$ (B): $\frac{1}{2}$ (C): 2 (D): 1 (E): None of the above.

Anwser. (D).

Question 4. A monkey in Zoo becomes lucky if he eats three different fruits. What is the largest number of monkeys one can make lucky, by having 20 oranges, 30 bananas, 40 peaches and 50 tangerines? Justify your answer.

(A): 30 (B): 35 (C): 40 (D): 45 (E): None of the above.

Anwser. (D).

Question 5. There are positive integers x, y such that $3x^2 + x = 4y^2 + y$, and $(x - y)$ is equal to

(A): 2013 (B): 2014 (C): 2015 (D): 2016 (E): None of the above.

Anwser. (E).

Question 6. Determine the smallest positive number a such that the number of all integers belonging to $(a, 2016a]$ is 2016.

Solution. The smallest integer greater than a is $[a]+1$ and the largest integer less than or is equal to $2016a$ is $[2016a]$. Hence, the number of all integers belonging to $(a, 2016a]$ is $[2016a]-[a]$.

Now we difine the smallest positive number a such that

$$[2016a]-[a]=2016.$$

If $0<a\leq 1$ then $[2016a]-[a]<2016$.

If $a\geq 2$ then $[2016a]-[a]>2016$.

Let $a=1+b$, where $0<b<1$. Then $[a]=1$, $[2016a]=2016+[2016b]$ and $[2016a]-[a]=2015+[2016b]=2016$ iff $[2016b]=1$. Hence the smallest positive number b such that $[2016b]=1$ is $b=\frac{1}{2016.}$

Thus, $a=1+\frac{1}{2016}$ is a smallest positive number such that the number of all integers belonging to $(a, 2016a]$ is 2016.

Question 7. Nine points form a grid of size 3×3. How many triangles are there with 3 vetices at these points?

Solution. We divide the triangles into two types:

Type 1: Two vertices lie in one horizontal line, the third vertice lies in another horizontal lines.

For this type we have 3 possibilities to choose the first line, 2 posibilities to choose 2nd line. In first line we have 3 possibilities to choose 2 vertices, in the second line we have 3 possibilities to choose 1 vertex. In total we have $3\times 2\times 3\times 3=54$ triangles of first type.

Type 2: Three vertices lie in distinct horizontal lines.

We have $3\times 3\times 3$ triangles of these type. But we should remove degenerated triangles from them. There are 5 of those (3 vertical lines and two diagonals). So, we have 27 - 5 = 22 triangles of this type.

Total we have 54 + 22 = 76 triangles.

For those students who know about C_n^k this problem can be also solved as C_9^3-8 where 8 is the number of degenerated triangles.

Question 8. Find all positive integers x, y, z such that

$$x^3-(x+y+z)^2=(y+z)^3+34.$$

Solution. Putting $y+z=a$, $a\in\mathbb{Z}$, $a\geq 2$, we have

$$x^3-a^3=(x+a)^2+34. \tag{1}$$

$$\Leftrightarrow (x-a)\left(x^2+xa+a^2\right)=x^2+2ax+a^2+34. \tag{2}$$

$$\Leftrightarrow (x-a-1)\left(x^2+xa+a^2\right)=xa+34.$$

Since x,a are integers, we have $x^2+xa+a^2\geq 0$ and $xa+34>0$. That follow $x-a-1>0$, i.e. $x-a\geq 2$.

This and (2) together imply

$$x^2+2ax+a^2+34\geq 2\left(x^2+xa+a^2\right)\Leftrightarrow x^2+a^2\leq 34.$$

Hence $x^2<34$ and $x<6$.

On the other hand, $x\geq a+2\geq 4$ then $x\in\{4,5\}$.

If $x=5$, then from $x^2+a^2\leq 34$ it follows $2\leq a\leq 3$. Thus $a\in\{2,3\}$.

The case of $x=5$, $a=2$ does not satisfy (1) for $x=5$, $a=3$, from (1) we find $y=1, z=2$ or $y=2, z=1$,

If $x=4$, then from the inequality $x-a\geq 2$ we find $a\leq 2$, which contradicts to (1).

Conclusion: $(x,y,z)=(5,1,2)$ and $(x,y,z)=(5,2,1)$.

Question 9. Let x,y,z satisfy the following inequalities

$$\begin{cases} |x+2y-3z|\leq 6 \\ |x-2y+3z|\leq 6 \\ |x-2y-3z|\leq 6 \\ |x+2y+3z|\leq 6 \end{cases}$$

Determine the greatest value of $M=|x|+|y|+|z|$.

Solution. Note that for all real numbers a,b,c, we have

$$|a|+|b|=\max\{|a+b|,|a-b|\}$$

and

$$|a|+|b|+|c|=\max\{|a+b+c|,|a+b-c|,|a-b-c|,|a-b+c|\}.$$

Hence

$$M=|x|+|y|+|z|\leq |x|+2|y|+3|z|=|x|+|2y|+|3z|$$

$$=\max\{|x+y+z|,|x+y-z|,|x-y-z|,|x-y+z|\}\leq 6.$$

Thus $\max M=6$ when $x=\pm 6$, $y=z=0$.

Question 10. Let h_a, h_b, h_c and r be the lengths of altitudes and radius of the inscribed circle of ΔABC, respectively. Prove that

$$h_a + 4h_b + 9h_c > 36r.$$

Solution. Let a, b, c be the side-lengths of ΔABC corresponding to h_a, h_b, h_c and S be the area of ΔABC. Then

$$ah_a = bh_b = ch_c = (a+b+c) \times r = 2S.$$

Hence

$$h_a + 4h_b + 9h_c = \frac{2S}{a} = \frac{8S}{b} = \frac{18S}{c} = 2S\left(\frac{1^2}{a} + \frac{2^2}{b} + \frac{3^2}{c}\right) \geq 2S\frac{(1+2+3)^2}{a+b+c}$$
$$= (a+b+c)\, r\frac{(1+2+3)^2}{a+b+c} = 36r.$$

The equality holds iff $a : b : c = 1 : 2 : 3$ (it is not posible for $a + b > c$).

Question 11. Let be given a triangle ABC and let I be the middle point of BC. The straight line d passing I intersects AB, AC at M, N, respectively. The straight line d' ($\not\equiv d$) passing I intersects AB, AC at Q and P, respectively. Suppose M, P are on the same side of BC and MP, NQ intersect BC at E and F, respectively. Prove that $IE = IF$.

Solution. Since $IB = IC$ then it is enough to show $\dfrac{EB}{EC} = \dfrac{FC}{FB}$.

By Menelaus theorem:

- For ΔABC and three points E, M, P, we have

$$\frac{EB}{EC} \times \frac{PC}{PA} \times \frac{MA}{MB} = 1$$

then

$$\frac{EB}{EC} = \frac{PA}{PC} \times \frac{MB}{MA}. \tag{1}$$

- For ΔABC and three points F, N, Q, we have

$$\frac{FC}{FB} \times \frac{QB}{QA} \times \frac{NA}{NC} = 1$$

then

$$\frac{FC}{FB} = \frac{NC}{NA} \times \frac{QA}{QB}. \tag{2}$$

- For ΔABC and three points M, I, N, we have

$$\frac{MB}{MA} \times \frac{NA}{NC} \times \frac{IC}{IB} = 1.$$

Compare with $IB = IC$ we find

$$\frac{MB}{MA} = \frac{NC}{NA}. \tag{3}$$

- For ΔABC and three points Q, I, P, we have

$$\frac{PA}{PC} \times \frac{IC}{IB} \times \frac{QB}{QA} = 1$$

then

$$\frac{PA}{PC} = \frac{QA}{QB}. \tag{4}$$

Equalities (1), (2), (3) and (4) toghether imply $IE = IF$.

Question 12. In the trapezoid $ABCD$, $AB \parallel CD$ and the diagonals intersect at O. The points P, Q are on AD, BC respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that $OP = OQ$.

Solution. Extending DA to B' such that $BB' = BA$, we find $\angle PB'B = \angle B'AB = \angle PDC$ and then triangles DPC and $B'PB$ are similar.

It follows that $\dfrac{DP}{PB'} = \dfrac{CD}{BB'} = \dfrac{CD}{BA} = \dfrac{DO}{BO}$ and so $PO \parallel BB'$.
Since triangles DPO and $DB'B$ are similar, we have

$$OP = BB' \times \frac{DO}{DB} = AB \times \frac{DO}{DB}.$$

Similarly, we have $OQ = AB \times \dfrac{CO}{CA}$ and it follows $OP = OQ$.

Question 13. Let H be orthocenter of the triangle ABC. Let d_1, d_2 be lines perpendicular to each-another at H. The line d_1 intersects AB, AC at D, E and the line d_2 intersects BC at F. Prove that H is the midpoint of segment DE if and only if F is the midpoint of segment BC.

Solution. Since $HD \perp HF, HA \perp FC$ and $HC \perp DA, \angle DAH = \angle HCF$ and $\angle DHA = \angle HFC$, therefore the triangles DHA, HFC are similar.

So $\dfrac{HA}{HD} = \dfrac{FC}{FH}$ (1)

Similarly, $\triangle EHA \backsim \triangle HFB$, so $\dfrac{HE}{HA} = \dfrac{FH}{FB}$ (2)

From (1) and (2), obtained $\dfrac{HE}{HD} = \dfrac{FC}{FB}$.
It follows H is midpoint of the segment DE iff F is midpoint of the segment BC.

Question 14. Given natural numbers a, b such that $2015a^2 + a = 2016b^2 + b$. Prove that $\sqrt{a-b}$ is a natural number.

Solution. From equality

$$2015a^2 + a = 2016b^2 + b, \tag{1}$$

we find $a \geq b$.

If $a = b$ then from (1) we have $a = b = 0$ and $\sqrt{a-b} = 0$.

If $a > b$, we write (1) as

$$b^2 = 2015(a^2 - b^2) + (a - b) \Leftrightarrow b^2 = (a - b)(2015a + 2015b + 1). \tag{2}$$

Let $(a, b) = d$ then $a = md$, $b = nd$, where $(m, n) = 1$. Since $a > b$ then $m > n$, and put $m - n = t > 0$.

Let $(t, n) = u$ then n is divisible by u, t is divisible by u and m is divisible by u. That follows $u = 1$ and then $(t, n) = 1$.

Putting $b = nd$, $a - b = td$ in (2), we find

$$n^2 d = t(2015dt + 4030dn + 1). \tag{3}$$

From (3) we get $n^2 d$ is divisible by t and compaire with $(t, n) = 1$, it follows d is divisible by t.

Also from (3) we get $n^2 d = 2015dt^2 + 4030dnt + t$ and then $t = n^2 d - 2015dt^2 - 4030dnt$.

Hence $t = d(n^2 - 2015t^2 - 4030nt)$, i.e. t is divisible by d, i.e. $t = d$ and then $a - b = td = d^2$ and $\sqrt{a-b} = d$ is a natural number.

Question 15. Find all polynomials of degree 3 with integer coefficients such that $f(2014) = 2015$, $f(2015) = 2016$, and $f(2013) - f(2016)$ is a prime number.

Solution. Let $g(x) = f(x) - x - 1$. Then $g(2014) = f(2014) - 2014 - 1 = 0$, $g(2015) = 2016 - 2015 - 1 = 0$. Hence $g(x) = (ax + b)(x - 2014)(x - 2015)$ and

$$f(x) = (ax + b)(x - 2014)(x - 2015) + x + 1, \;\; a, b \in \mathbb{Z}, a \neq 0.$$

We have $f(2013) = 2(2013a + b) + 2014$ and

$$f(2016) = 2(2016a + b) + 2017.$$

That follows

$$\begin{aligned} f(2013) - f(2016) &= 2(2013a + b) + 2014 - [2(2016a + b) + 2017] \\ &= -6a - 3 = 3(-2a - 1) \end{aligned}$$

and $f(2013) - f(2016)$ is prime iff $-2a - 1 = 1$, i.e. $a = -1$.

Conlusion: All polynomials of degree 3 with integer coefficients such that $f(2014) = 2015$, $f(2015) = 2016$ and $f(2013) - f(2016)$ is a prime number are of the form

$$f(x) = (b - x)(x - 2014)(x - 2015) + x + 1, \;\; b \in \mathbb{Z}.$$