Hanoi Open Mathematical Competition 2017

Junior Section

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial

$$P(x) = x^3 - 6x^2 + 5x + 12.$$

The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 14 (E): None of the above.

Solution. The choice is (C).

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 1?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Solution. The choice is (A).

Question 3. Suppose $n^2+4n+25$ is a perfect square. How many such non-negative integers n's are there?
(A): 1 (B): 2 (C): 4 (D): 6 (E): None of the above.

Solution. The choice is (B).

Question 4. Put

$$S = 2^1 + 3^5 + 4^9 + 5^{13} + \cdots + 505^{2013} + 506^{2017}.$$

The last digit of S is
(A): 1 (B): 3 (C): 5 (D): 7 (E): None of the above.

Solution. The choice is (E).

Question 5. Let a, b, c be two-digit, three-digit, and four-digit numbers, respectively. Assume that the sum of all digits of number $a + b$, and the sum of all digits of $b + c$ are all equal to 2. The largest value of $a + b + c$ is
(A): 1099 (B): 2099 (C): 1199 (D): 2199 (E): None of the above.

Solution. The choice is (E).

Question 6. Find all triples of positive integers (m, p, q) such that

$$2^m p^2 + 27 = q^3, \quad \text{and } p \text{ is a prime.}$$

Solution. By the assumption it follows that q is odd. We have

$$2^m p^2 = (q-3)(q^2+3q+9).$$

Remark that q^2+3q+9 is always odd. There are two cases:
Case 1. $q = 2^m p + 3$. We have

$$q^3 = (2^m p+3)^3 > 2^m p^2 + 27,$$

which is impossible.

Case 2. $q = 2^m + 3$. We have

$$q^3 = 2^{3m} + 9 \times 2^{2m} + 27 \times 2^m + 27 = 2^m p^2 + 27,$$

which implies

$$p^2 = 2^{2m} + 9 \times 2^m + 27.$$

If $m \geq 3$, then $2^{2m} + 9 \times 2^m + 27 \equiv 3 \pmod 8$, but $p^2 \equiv 1 \pmod 8$. We deduce $m \leq 3$. By simple computation we find $m = 1, p = 7, q = 5$.

Question 7. Determine two last digits of number

$$Q = 2^{2017} + 2017^2.$$

Solution. We have

$$\begin{aligned} 2^{2017} &= 2^7 \times \left(2^{10}\right)^{201} = 128 \times 1024^{201} \\ &\equiv 128 \times (-1)^{201} = -128 \equiv 22 \pmod{25}; \\ 2017^2 &\equiv 14 \pmod{25}. \end{aligned}$$

It follows $P \equiv 11 \pmod{25}$, by which two last digits of P are in the set $\{11, 36, 61, 86\}$. In other side, $P \equiv 1 \pmod 4$. This implies $P \equiv 61 \pmod{100}$. Thus, the number 61 subjects to the question.

Question 8. Determine all real solutions x, y, z of the following system of equations

$$\begin{cases} x^3 - 3x &= 4 - y \\ 2y^3 - 6y &= 6 - z \\ 3z^3 - 9z &= 8 - x. \end{cases}$$

Solution. From $x^3 + y = 3x + 4$ it follows $x^3 - 2 - 3x = 2 - y$. Then

$$(x-2)(x+1)^2 = 2 - y \tag{1}$$

By $2y^3 - 4 - 6y = 2 - z$, we have

$$2(y-2)(y+1)^2 = 2 - z. \tag{2}$$

Similarly, by $3z^3 - 3 - 3 - 9z = 2 - x$ we have

$$3(z-2)(z+1)^2 = (2 - x). \tag{3}$$

Combining (1)-(2)-(3) we obtain

$$(x-2)(y-2)(z-2)\Big((x+1)^2(y+1)^2(z+1)^2 + \frac{1}{6}\Big) = 0.$$

Hence, $(x-2)(y-2)(z-2) = 0$. Comparing this with (1), (2) and (3), we find the unique solution $x = y = z = 2$.

Question 9. Prove that the equilateral triangle of area 1 can be covered by five arbitrary equilateral triangles having the total area 2.

Solution.

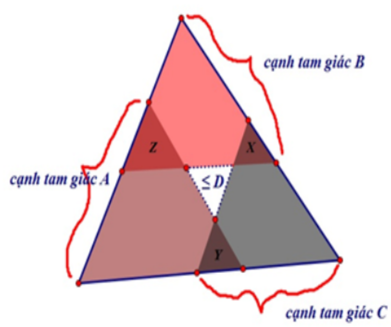

Figure 1: For Question 9

Let S denote the triangle of area 1. It is clearly that if $a \geq b$ then triangle of area a can cover triangle of area b. It suffices to consider the case when the areas of five small triangles are all smaller than 1. Let $1 \geq A \geq B \geq C \geq D \geq E$ stand for the areas. We will prove that the sum of side-lengths of B and C is not smaller than the side-length of triangle of area 1. Indeed, suppose $\sqrt{B} + \sqrt{C} < \sqrt{1} = 1$. It follows

$$2 = A + B + C + D + E < 1 + B + C + 2\sqrt{BC} = 1 + (\sqrt{B} + \sqrt{C})^2 < 2,$$

which is impossible.

We cover S by A, B, C as Figure 1. We see that A, B, C will have common parts, mutually. Suppose

$$X = B \cap C; \quad Y = A \cap C; \quad Z = A \cap B.$$

It follows

$$X + Y \leq C; \quad Y + Z \leq A; \quad Z + X \leq B.$$

We deduce A, B, C cover a part of area:

$$\begin{aligned} A + B + C - X - Y - Z &\geq A + B + C - \frac{1}{2}\big[(X+Y) + (Y+Z) + (Z+X)\big] \\ &\geq \frac{1}{2}(A + B + C) = 1 - \frac{D+E}{2} \geq 1 - D. \end{aligned}$$

Thus, D can cover the remained part of S.

Question 10. Find all non-negative integers a, b, c such that the roots of equations:

$$x^2 - 2ax + b = 0; \quad (1)$$
$$x^2 - 2bx + c = 0; \quad (2)$$
$$x^2 - 2cx + a = 0 \quad (3)$$

are non-negative integers.

Solution. We see that $a^2 - b, b^2 - c, c^2 - a$ are perfect squares. Namely,

$$a^2 - b = p^2; \quad b^2 - c = q^2; \quad c^2 - a = r^2.$$

There are two cases:
Case 1. $b = 0$. We derive that $b = c = 0$. Thus $(a, b, c) = (0, 0, 0)$ is unique solution.
Case 2. $a, b, c \neq 0$. We have $a^2 - b \leq (a-1)^2 = a^2 - 2a + 1$. This implies $b \geq 2a - 1$. Similarly, we can prove that $c \geq 2b - 1$, and $a \geq 2c - 1$. Combining three above inequalities we deduce $a + b + c \leq 3$. By simple computation we obtain $(a, b, c) = (1, 1, 1)$.

Question 11. Let S denote a square of the side-length 7, and let eight squares of the side-length 3 be given. Show that S can be covered by those eight small squares.

Solution.
Figure 2 is a solution.

Question 12. Does there exist a sequence of 2017 consecutive integers which contains exactly 17 primes?

Solution. It is easy to see that there are more than 17 primes in the sequence of numbers $1, 2, 3, 4, \ldots, 2017$. Precisely, there are 306 primes in that sequence. Remark that if the sequence

$$k + 1, k + 2, \ldots, k + 2017$$

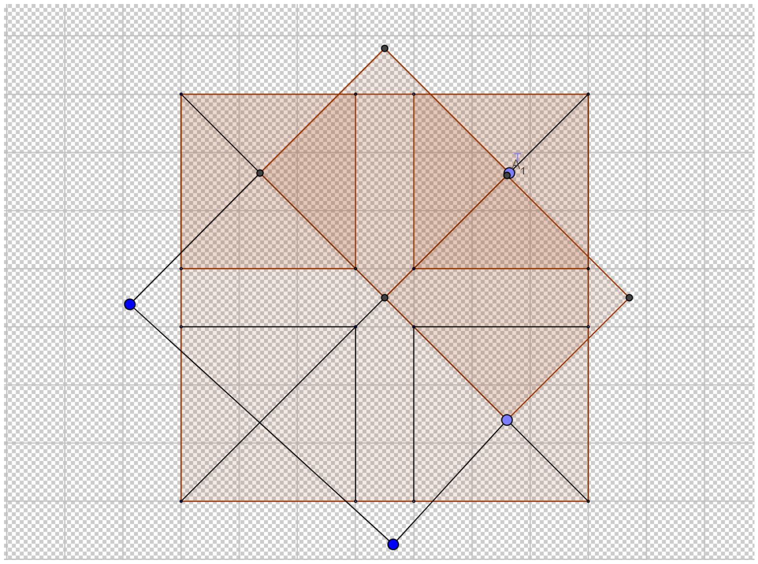

Figure 2: For Question 11

was changed by the sequence

$$k, k+1, \ldots, k+2016,$$

then the numbers of primes in the latter and former sequences are either equal, more or less by 1. In what follows, we say such change *a back-shift with 1-step.* First moment, we consider the sequence of 2017 consecutive integers:

$$2018!+2, 2018!+3, \ldots 2018!+2018$$

which contain no prime. After 2018!+1 times back-shifts with 1-step, we obtain the sequence

$$1, 2, 3, 4, \ldots, 2017.$$

The last sequence has 306 primes, while the first sequence has no prime. Reminding the above remark we conclude that there is a moment in which the sequence contains exactly 17 primes.

Question 13. Let a, b, c be the side-lengths of triangle ABC with $a+b+c=12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Solution. Put

$$x := \frac{b+c-a}{2}, y := \frac{c+a-b}{2}, z := \frac{a+b-c}{2}.$$

Then $x, y, z > 0$, and

$$x+y+z = \frac{a+b+c}{2} = 6,\ a = y+z, b = z+x, c = x+y.$$

We have

$$M = \frac{y+z}{2x} + \frac{4(z+x)}{2y} + \frac{9(x+y)}{2z} = \frac{1}{2}\left[\left(\frac{y}{x} + \frac{4x}{y}\right) + \left(\frac{z}{x} + \frac{9x}{z}\right) + \left(\frac{4z}{y} + \frac{9y}{z}\right)\right]$$

$$\geq \frac{1}{2}\left(2\sqrt{\frac{y}{x}.\frac{4x}{y}} + 2\sqrt{\frac{z}{x}.\frac{9x}{z}} + 2\sqrt{\frac{4z}{x}.\frac{9y}{z}}\right) = 11.$$

The equality occurs in the above if and only if

$$\begin{cases} \dfrac{y}{x} = \dfrac{4x}{y} \\ \dfrac{z}{x} = \dfrac{9x}{z} \\ \dfrac{4z}{y} = \dfrac{9y}{z}, \end{cases}$$

or

$$\begin{cases} y = 2x \\ z = 3x \\ 2z = 3y. \end{cases}$$

Since $x+y+z=6$ we receive $x=1,\ y=2,\ z=3$. Thus $\min S = 11$ if and only if $(a,b,c) = \big(5,4,3\big)$.

Question 14. Given trapezoid $ABCD$ with bases $AB \parallel CD$ ($AB < CD$). Let O be the intersection of AC and BD. Two straight lines from D and C are perpendicular to AC and BD intersect at E, i.e. $CE \perp BD$ and $DE \perp AC$. By analogy, $AF \perp BD$ and $BF \perp AC$. Are three points E, O, F located on the same line?

Solution. Since E is the orthocenter of triangle ODC, and F is the orthocenter of triangle OAB we see that OE is perpendicular to CD, and OF is perpendicular to AB. As AB is parallel to CD, we conclude that E, O, F are straightly lined.

Question 15. Show that an arbitrary quadrilateral can be divided into nine isosceles triangles.

Solution. Remark that a quadrilateral can be divided into two triangles. Therefore, it suffices to cut an arbitrary triangle into for and five isosceles triangles. Figures 3, 4, and 5 shows some solution.

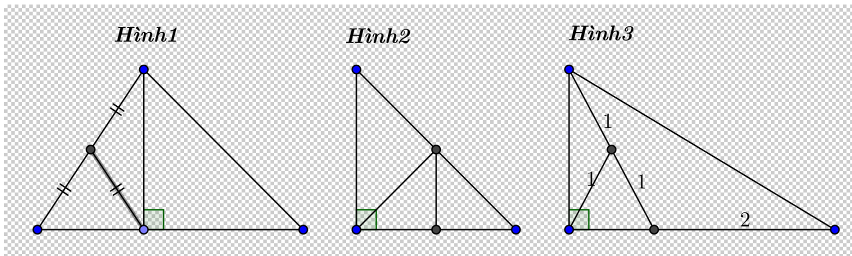

Figure 3: For Question 15

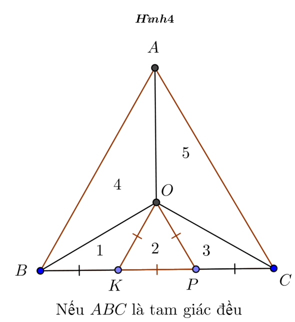

Figure 4: For Question 15

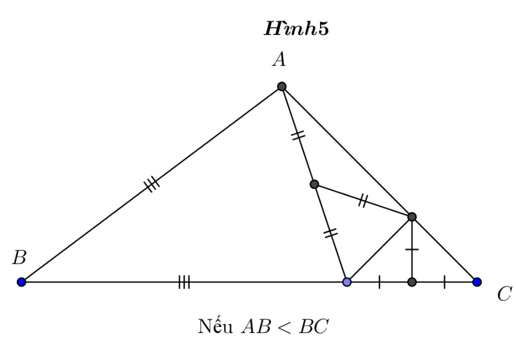

Figure 5: For Question 15