Hanoi Open Mathematical Competition 2015

Senior Section

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. The sum of all even positive intergers less than 100 those are not divisible by 3 is

(A): 938; (B): 940; (C): 1634; (D): 1638; (E): None of the above.

Answer: C.

Question 2. A regular hexagon and an equilateral triangle have equal perimeter. If the area of the triangle is $4\sqrt{3}$ square units, the area of the hexagon is

(A): $5\sqrt{3}$; (B): $6\sqrt{3}$; (C): $7\sqrt{3}$; (D): $8\sqrt{3}$; (E): None of the above.

Answer: B.

Question 3. Suppose that $a > b > c > 1$. One of solutions of the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = x$$

is

(A): -1; (B): -2; (C): 0; (D): 1; (E): None of the above.

Answer: D.

Question 4. Let a, b, c and m $(0 \leq m \leq 26)$ be integers such that

$$a + b + c = (a-b)(b-c)(c-a) = m \pmod{27}$$

then m is

(A): 0; (B): 1; (C): 25; (D): 26; (E): None of the above.

Answer: A.

Question 5. The last digit of number $2017^{2017} - 2013^{2015}$ is

(A): 2; (B): 4; (C): 6; (D): 8; (E): None of the above.

Answer: E.

Question 6. Let $a, b, c \in [-1, 1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that

$$1 + 2a^2b^2c^2 \geq a^4 + b^4 + c^4.$$

Solution. The constraint can be written as

$$(a - bc)^2 \leq (1 - b^2)(1 - c^2). \tag{1}$$

Using the Cauchy inequality, we have

$$(a + bc)^2 \leq (|a| + |bc|)^2 \leq (1 + |b||c|)^2 \leq (1 + b^2)(1 + c^2).$$

Multiplying by (1), we get

$$\begin{aligned} (a - bc)^2(a + bc)^2 &\leq (1 - b^2)(1 + b^2)(1 - c^2)(1 + c^2) \\ \Leftrightarrow (a^2 - b^2c^2)^2 &\leq (1 - b^2)(1 + b^2)(1 - c^2)(1 + c^2) \\ \Leftrightarrow (a^2 - b^2c^2)^2 &\leq (1 - b^4)((1 - c^4) \\ \Leftrightarrow 1 + 2a^2b^2c^2 &\geq a^4 + b^4 + c^4. \end{aligned}$$

Question 7. Solve equation

$$x^4 = 2x^2 + [x], \tag{2}$$

where $[x]$ is an integral part of x.

Solution.

We have

$$(2) \Leftrightarrow [x] = x^2\left(x^2 - 2\right)$$

Consider the case $x^2 \leq 2$, then $-\sqrt{2} \leq x \leq \sqrt{2}$ and $[x] \leq 0$. It follows $[x] \in \{-1; 0\}$.

If $[x] = 0$, then from (2) we find $x = 0$ as a solution.

If $[x] = -1$, then from (2) we find $x = -1$ as a solution.

Now we suppose that $x^2 > 2$. It follows from (2), $[x] > 0$ and then $x > \sqrt{2}$. Hence $x^2\left(x^2 - 2\right) = \dfrac{[x]}{x} \leq 1$ and $x^2 - 2 \leq \dfrac{1}{x} < 1$. It follows $x < \sqrt{3}$, i.e. $\sqrt{2} < x < \sqrt{3}$.

It means that $[x] = 1$ and then $x = \sqrt{1 + \sqrt{2}}$ is a solution of the equation.

Question 8. Solve the equation

$$(x + 1)^3(x - 2)^3 + (x - 1)^3(x + 2)^3 = 8(x^2 - 2)^3. \tag{3}$$

Solution. Rewrite equation (1) in the form

$$(x^2 - x - 2)^3 + (x^2 + x - 2)^3 = (2x^2 - 4)^3. \tag{4}$$

Factoring the sum of cubes on the right side of the equation (1), we find that one factor is $(2x^2-4)$, thus, two solutions of the equation is $x = \pm\sqrt{2}$.

Now we rewrite the equation (2) as

$$(x^2-x-2)^3 = (2x^2-4)^3 - (x^2+x-2)^3. \tag{5}$$

Factoring the difference of cubes on the left side of the equation (), we find that one factor is (x^2-x-2), thus, two solution of the equation is $x=-1, x=2$.

Finally, we rewrite (4) as

$$(x^2+x-2)^3 = (2x^2-4)^3 - (x^2-x-2)^3.$$

Factoring again, we see that one factor is (x^2+x-2). Thus, two solutions of the equation are $x=1, x=-2$..

Since the left hand side of the equation $8(x^2-2)^3-(x+1)^3(x-2)^3-(x-1)^3(x+2)^3=0$ is a polynomial of degree 6, then it has at most 6 roots, and we have them.

Hence,

$$8(x^2-2)^3-(x+1)^3(x-2)^3-(x-1)^3(x+2)^3 = 6(x^2-2)(x^2-1)(x^2-4).$$

Question 9. Let a, b, c be positive numbers with $abc = 1$. Prove that

$$a^3+b^3+c^3+2[(ab)^3+(bc)^3+(ca)^3] \geq 3(a^2b+b^2c+c^2a).$$

Solution. Asume that $a = \max\{a,b,c\}$ then $a \geq b \geq c > 0$ or $a \geq c \geq b > 0$ and

$$a^3+b^3+c^3-(a^2b+b^2c+c^2a) = (a-b)(a^2-c^2)+(b-c)(b^2-c^2) \geq 0.$$

Hence

$$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a. \tag{6}$$

Since $\frac{1}{c} = \max\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ or $\frac{1}{b} = \max\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$, then

$$\frac{1}{c^3}+\frac{1}{b^3}+\frac{1}{a^3} \geq \frac{1}{c^2b}+\frac{1}{b^2a}+\frac{1}{a^2c}.$$

Since $abc = 1$, this can be written as

$$(ab)^3+(bc)^3+(ca)^3 \geq a^2b+b^2c+c^2a. \tag{7}$$

(6) and (7) together imply the proposed inequality.

Question 10. A right-angled triangle has property that, when a square is drawn externally on each side of the triangle, the six vertices of the squares that are not vertices of the triangle are concyclic. Assume that the area of the triangle is 9 cm^2. Determine the length of sides of the triangle.

Solution. We have $OJ = OD = OG =$ radius of the circle. Let the sides of ΔABC be a, b, c. Then

$$OJ^2 = OM^2 + MJ^2 = \left(b + \frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = b^2 + ab + \frac{1}{4}(a^2 + b^2).$$

$$OD^2 = ON^2 + ND^2 = \left(a + \frac{b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = a^2 + ab + \frac{1}{4}(a^2 + b^2).$$

$$OG^2 = OL^2 + LG^2 = c^2 + \left(\frac{c}{2}\right)^2 = \frac{5}{4}c^2 = \frac{5}{4}(a^2 + b^2) = a^2 + b^2 + \frac{1}{4}(a^2 + b^2).$$

Comparing these right-hand sides, we get

$$b^2 + ab = a^2 + ab = a^2 + b^2 \Leftrightarrow a = b.$$

It means that the given triangle with the desired property is the isosceles right triangle and then $\frac{1}{2}ab = 9 \Leftrightarrow a = b = 3\sqrt{2}, c = 6$ units.

Question 11. Given a convex quadrilateral $ABCD$. Let O be the intersection point of diagonals AC and BD and let I, K, H be feet of perpendiculars from B, O, C to AD, respectively. Prove that

$$AD \times BI \times CH \leq AC \times BD \times OK.$$

Solution. Draw $AE \perp BD$ $(E \in BD)$.

We have $S_{ABD} = \dfrac{BI \times AD}{2} = \dfrac{AE \times BD}{2}$. Then $BI \times AD = AE \times BD$. It follows $BI \times AD \leq AO \times BD$ $(AE \leq AO)$ and $BI.AD \leq AC \times BD \times \dfrac{AO}{AC}$.

Moreover, we have $OK \parallel CH$ then $\dfrac{AO}{AC} = \dfrac{OK}{CH}$ and $BI \times AD \leq AC \times BD \times \dfrac{OK}{CH}$. It follows $BI \times AD \times CH \leq AC \times BD \times OK$.

The equality holds if quadrilateral $ABCD$ has two perpendicular diagonals.

Question 12. Give an isosceles triangle ABC at A. Draw ray Cx being perpendicular to CA, BE perpendicular to Cx $(E \in Cx)$. Let M be the midpoint of BE, and D be the intersection point of AM and Cx. Prove that BD $\perp$ BC

Solution. Let K be intersection point of DB and AC.

Since $BE \perp CD$; $CK \perp CD$ then $BE \parallel CK$.

In ΔDAC we see $ME \parallel AC$ so

$$\frac{ME}{AC} = \frac{DM}{AD} \tag{8}$$

In ΔDAK we see $MB \parallel AK$ so

$$\frac{MB}{AK} = \frac{DM}{AD}. \tag{9}$$

From (8) and (9), we get $\frac{ME}{AC} = \frac{MB}{AK}$. This and equality $MB = ME$ together imply

$AK = AC = AB$ and then $BA = \frac{CK}{2}$.

Note that BA is a median line ΔBKC and $BA = \frac{CK}{2}$ then ΔBKC is a right triangle at B. Hence $BD \perp BC$.

Question 13. Let m be given odd number, and let a, b denote the roots of equation $x^2 + mx - 1 = 0$ and $c = a^{2014} + b^{2014}$, $d = a^{2015} + b^{2015}$. Prove that c and d are relatively prime numbers.

Solution. Since $a^2 + ma - 1 = 0$ then $a \neq 0$ and

$$a^{n+2} = -ma^{n+1} + a^n \quad \forall n \in \mathbb{N}.$$

Similarly, $b^{n+2} = -mb^{n+1} + b^n; \ \ \forall n \in \mathbb{N}$.

Hence, the sequence x_n, $n \in \mathbb{N}$ are defined as

$$\begin{cases} x_0 = 2, \ x_1 = -m \\ x_{n+2} = -mx_{n+1} + x_n \quad \forall n \in \mathbb{N}. \end{cases}$$

It is easy to see all x_n are integers. Hence, c, d are integers.

Now we prove $(x_n; x_{n+1}) = 1$ for every $n \in \mathbb{N}$.

For $n = 0$, $x_0 = 2$ and m is odd then $(x_0; x_1) = (2; -m) = 1$.

Suppose that $(x_k; x_{k+1}) = 1$ for $k \geq 0$ and $(x_{k+1}; x_{k+2}) > 1$. Let p be a prime factor of x_{k+1} and x_{k+2}, then from $x_k = x_{k+2} + mx_{k+1}$, it follows p is a prime factor of x_k. It means that $p \mid (x_k; x_{k+1}) = 1$, absurd. Hence $(x_{k+1}; x_{k+2}) = 1$ and $(c, d) = 1$.

Question 14. Determine all pairs of integers $(x; y)$ such that

$$2xy^2 + x + y + 1 = x^2 + 2y^2 + xy.$$

Solution. We have

$$2xy^2 + x + y + 1 = x^2 + 2y^2 + xy$$

$$\Leftrightarrow 2y^2(x-1) - y(x-1) - x(x-1) = -1$$
$$\Leftrightarrow (x-1)(2y^2 - y - x) = -1$$

Since $x; y$ are integers then $x - 1$ and $2y^2 - y - x$ are divisors of -1.

Case 1. $$\begin{cases} x - 1 = 1 \\ 2y^2 - y - x = -1 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x = 2 \\ y = 1 \end{cases} \\ \begin{cases} x = 2 \\ y = -\frac{1}{2} \text{(absurd)} \end{cases} \end{array} \right.$$

- Case 2. $\begin{cases} x-1=-1 \\ 2y^2-y-x=1 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x=0 \\ y=1 \end{cases} \\ \begin{cases} x=0 \\ y=-\dfrac{1}{2} \text{(absurd)} \end{cases} \end{array} \right.$

Hence all integral pairs $(x; y)$ are $(2;\ 1)$; $(0\ ;\ 1)$.

Question 15. Let the numbers a, b, c, d satisfy the relation $a^2 + b^2 + c^2 + d^2 \leq 12$. Determine the maximum value of

$$M = 4(a^3 + b^3 + c^3 + d^3) - (a^4 + b^4 + c^4 + d^4).$$

Solution. Note that $x^2(x-2)^2 \geq 0$ for each real x. This inequality can be rewritten as $4x^3 - x^4 \leq 4x^2$. It follows that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \leq 4(a^2 + b^2 + c^2 + d^2) = 48,$$

The equality holds for $(a, b, c, d) = (2, 2, 2, 0).(2, 2, 0, 2), (2, 0, 2, 2), (0, 2, 2, 2)$. Hence, $\max M = 48$.