Hanoi Open Mathematical Competition 2017

Senior Section

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. Suppose x_1, x_2, x_3 are the roots of polynomial $P(x) = x^3 - 4x^2 - 3x + 2$. The sum $|x_1| + |x_2| + |x_3|$ is
(A): 4 (B): 6 (C): 8 (D): 10 (E): None of the above.

Solution. The solution is (B).

Question 2. How many pairs of positive integers (x, y) are there, those satisfy the identity

$$2^x - y^2 = 4?$$

(A): 1 (B): 2 (C): 3 (D): 4 (E): None of the above.

Solution. The solution is (A).

Question 3. The number of real triples (x, y, z) that satisfy the equation

$$x^4 + 4y^4 + z^4 + 4 = 8xyz$$

is
(A): 0; (B): 1; (C): 2; (D): 8; (E): None of the above.

Solution. The solution is (E).

Question 4. Let a, b, c be three distinct positive numbers. Consider the quadratic polynomial

$$P(x) = \frac{c(x-a)(x-b)}{(c-a)(c-b)} + \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + 1.$$

The value of $P(2017)$ is
(A): 2015 (B): 2016 (C): 2017 (D): 2018 (E): None of the above.

Solution. The solution is (D).

Question 5. Write 2017 following numbers on the blackboard:

$$-\frac{1008}{1008}, -\frac{1007}{1008}, \ldots, -\frac{1}{1008}, 0, \frac{1}{1008}, \frac{2}{1008}, \ldots, \frac{1007}{1008}, \frac{1008}{1008}.$$

One processes some steps as: erase two arbitrary numbers x, y on the blackboard and then write on it the number $x + 7xy + y$. After 2016 steps, there is only one number. The last one on the blackboard is

(A): $-\dfrac{1}{1008}$ (B): 0 (C): $\dfrac{1}{1008}$ (D): $-\dfrac{144}{1008}$ (E): None of the above.

Solution. The solution is (D).

Question 6. Find all pairs of integers a, b such that the following system of equations has a unique integral solution (x, y, z)

$$\begin{cases} x + y = a - 1 \\ x(y+1) - z^2 = b. \end{cases}$$

Solution. Write the given system in the form

$$\begin{cases} x + y + 1 = a \\ x(y+1) - z^2 = b. \end{cases} \qquad (*)$$

System (*) is symmetric by $x, y+1$ and is reflect in z at 0 then the necessary condition for (*) to have a unique solution is $(x, y+1, z) = (t, t, 0)$. Putting this in (*), we find $a^2 = 4b$. Conversely, if $a^2 = 4b$ then

$$(x - (y+1))^2 + 4z^2 = (x + y + 1)^2 + 4z^2 - 4x(y+1) = a^2 - 4b = 0.$$

This implies the system has a unique solution

$$(x, y+1, z) = \left(\frac{a}{2}, \frac{a}{2}, 0\right).$$

Question 7. Let two positive integers x, y satisfy the condition $x^2 + y^2 \vdots 44$. Determine the smallest value of $T = x^3 + y^3$.

Solution. By the assumption we have $x^2 + y^2 \vdots 11$. One can prove that $x \vdots 11$ and $y \vdots 11$. Due to limitted space we left the proof for the reader. In other side, by the assumption we have x and y are even. Hence, $x \equiv 0 \pmod{22}$ and $y \equiv 0 \pmod{22}$. Thus, $\min A = (22)^3 + (22)^3 = 21296$.

Question 8. Let a, b, c be the side-lengths of triangle ABC with $a + b + c = 12$. Determine the smallest value of

$$M = \frac{a}{b+c-a} + \frac{4b}{c+a-b} + \frac{9c}{a+b-c}.$$

Solution. Let $x = \dfrac{b+c-a}{2}, y = \dfrac{c+a-b}{2}, z = \dfrac{a+b-c}{2}$ then $x, y, z > 0$ and

$x+y+z=\dfrac{a+b+c}{2}=6, a=y+z, b=z+x, c=x+y$. We have

$$M=\frac{y+z}{2x}+\frac{4(z+x)}{2y}+\frac{9(x+y)}{2z}=\frac{1}{2}\left[\left(\frac{y}{x}+\frac{4x}{y}\right)+\left(\frac{z}{x}+\frac{9x}{z}\right)+\left(\frac{4z}{y}+\frac{9y}{z}\right)\right]$$
$$\geq\frac{1}{2}\left(2\sqrt{\frac{y}{x}.\frac{4x}{y}}+2\sqrt{\frac{z}{x}.\frac{9x}{z}}+2\sqrt{\frac{4z}{y}.\frac{9y}{z}}\right)=11.$$

The equality yields if and only if

$$\begin{cases}\dfrac{y}{x}=\dfrac{4x}{y}\\ \dfrac{z}{x}=\dfrac{9x}{z}\\ \dfrac{4z}{y}=\dfrac{9y}{z}.\end{cases}$$

Equivelently,

$$\begin{cases}y=2x\\ z=3x\\ 2z=3y.\end{cases}$$

By simple computation we receive $x=1, y=2, z=3$. Therefore, $\min S=11$ when $(a,b,c)=\big(5,4,3\big)$.

Question 9. Cut off a square carton by a straight line into two pieces, then cut one of two pieces into two small pieces by a straight line, ect. By cutting 2017 times we obtain 2018 pieces. We write number 2 in every triangle, number 1 in every quadrilateral, and 0 in the polygons. Is the sum of all inserted numbers always greater than 2017?

Solution. After 2017 cuts, we obtain 2018 n-convex polygons with $n\geq 3$. After each cut the total of all sides of those n-convex polygons increases at most 4. We deduce that the total number of sides of 2018 pieces is not greater than 4×2018. If the side of a piece is k_j, then the number inserted on it is greater or equal to $5-k_j$. Therefore, the total of all inserted numbers on the pieces is greater or equal to

$$\sum_j(5-k_j)=5\times 2018-\sum_j k_j\geq 5\times 2018-4\times 2018=2018>2017.$$

The answer is positive.

Question 10. Consider all words constituted by eight letters from $\{C,H,M,O\}$. We arrange the words in an alphabet sequence. Precisely, the first word is CCCC-CCCC, the second one is CCCCCCCH, the third is CCCCCCCM, the fourth one is CCCCCCCO,..., and the last word is OOOOOOOO.

a) Determine the 2017^{th} word of the sequence?

b) What is the position of the word HOMCHOMC in the sequence?

Solution. We can associate the letters C, H, M, O with four numbers $0, 1, 2, 3$, respectively. Thus, the arrangement of those words as a dictionary is equivalent to arrangement of those numbers increasing.

a) Number 2017 in quaternary is $\{133201\}_4 = \{00133201\}_4 \sim CCHOOMCH$.

b) The word $HOMCHOMC$ is corresponding to the number $\{13201320\}_4$ which means the number 13201320 in quaternary. Namely,

$$\{13201320\}_4 = 4^7 + 3 \times 4^6 + 2 \times 4^5 + 0 \times 4^4 + 1 \times 4^3 + 3 \times 4^2 + 2 \times 4 + 0.$$

A simple computation gives $\{13201320\}_4 = 30840$. Thus, the word $HOMCHOMC$ is 30840^{th} in the sequence.

Question 11. Let ABC be an equilateral triangle, and let P stand for an arbitrary point inside the triangle. Is it true that

$$\left|\widehat{PAB} - \widehat{PAC}\right| \geq \left|\widehat{PBC} - \widehat{PCB}\right|?$$

Solution. If P lies on the symmetric straightline Ax of ΔABC, then

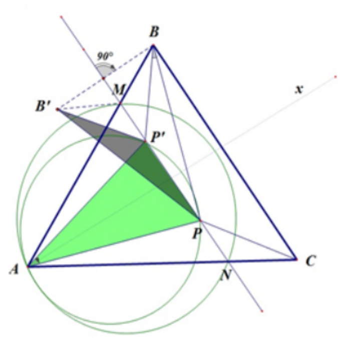

Figure 1: For Question 11

$$\left|\widehat{PAB} - \widehat{PAC}\right| = \left|\widehat{PBC} - \widehat{PCB}\right|.$$

We should consider other cases. Let P' denote the symmetric point of P with respect to Ax. The straightline PP'' intersects AB and AC at M and N, respectively. Choose B' that is symmetric point of B with respect to MN. Then

$$\left|\widehat{PAB} - \widehat{PAC}\right| = \widehat{PAP'},$$

and

$$\left|\widehat{PBC} - \widehat{PCB}\right| = \widehat{PBP'} = \widehat{PB'P'}.$$

We will prove that

$$\widehat{PAP'} \geq \widehat{PB'P'}. \tag{$*$}$$

Indeed, consider the circumscribed circle (O) of the equilateral triangle AMN. Since

$$\widehat{MB'N} = \widehat{MBN} \leq \widehat{MBC} = \widehat{MAN} = 60^0,$$

B' is outside (O). Consider the circumscribed circle (O') of the equilateral triangle APP'. It is easy to see that (O') inside (O), by which B' is outside (O'). Hence, $\widehat{PAP'} \geq \widehat{PB'P'}$. The inequality (*) is proved.

Question 12. Let (O) denote a circle with a chord AB, and let W be the midpoint of the minor arc AB. Let C stand for an arbitrary point on the major arc AB. The tangent to the circle (O) at C meets the tangents at A and B at points X and Y, respectively. The lines WX and WY meet AB at points N and M, respectively.

Does the length of segment NM depend on position of C?

Solution. Let T be the common point of AB and CW. Consider circle (Q) touching XY at C and touching AB at T.

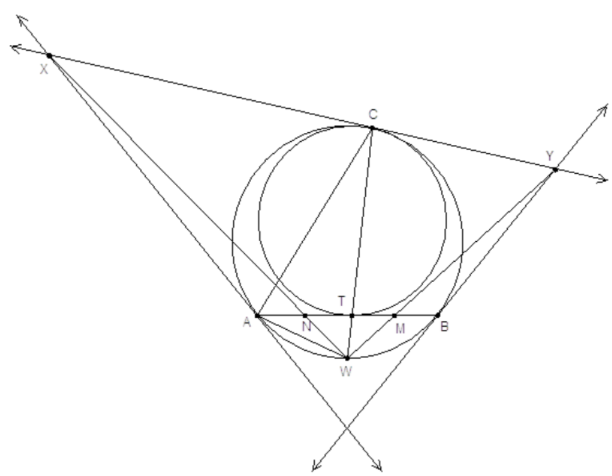

Figure 2: For Question 12

Since

$$\widehat{ACW} = \widehat{WAT} \quad \left(= \frac{1}{2}\overset{\frown}{AW} = \frac{1}{2}\overset{\frown}{WB}\right)$$

and $\widehat{AWT} = \widehat{CWA}$, we obtain that $\Delta AWT, \Delta CWA$ are similar triangles. Then

$$WA^2 = WT \times WC.$$

It is easy to see that WX is the radical axis of A and (Q), thus it passes through the midpoint N of segment AT. Similarly, WY passes through the midpoint M of segment BT. We deduce $MN = \dfrac{AB}{2}$.

Question 13. Let ABC be a triangle. For some $d > 0$ let P stand for a point inside the triangle such that

$$|AB| - |PB| \geq d, \text{ and } |AC| - |PC| \geq d.$$

Is the following inequality true

$$|AM| - |PM| \geq d,$$

for any position of $M \in BC$?

Solution. Note that AM always intersects PB or PC of ΔPBC. Without loss of generality, assume that AM has common point with PB. Then $ABMP$ is a convex quadrilateral with diagonals AM and PB.

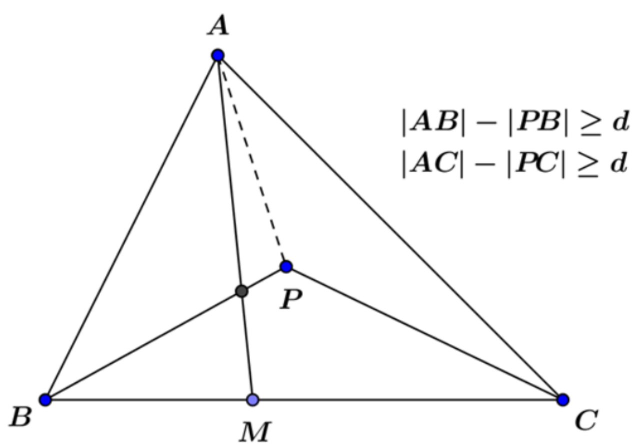

Figure 3: For Question 13

It is known that for every convex quadrilateral, we have

$$|AM| + |PB| \geq |AB| + |PM|,$$

that follows

$$|AM| - |PM| \geq |AB| - |PB| \geq d.$$

Question 14. Put

$$P = m^{2003}n^{2017} - m^{2017}n^{2003}, \quad \text{where} \quad m, n \in \mathbb{N}.$$

a) Is P divisible by 24?
b) Do there exist $m, n \in \mathbb{N}$ such that P is not divisible by 7?

Solution. We have

$$P = m^{2003}n^{2013}\left(n^{14} - m^{14}\right) = m^{2003}n^{2013}\left(n^7 - m^7\right)\left(n^7 + m^7\right).$$

It is easy to prove P is divisible by 8, and by 3.

b) It suffices to chose m, n such that the remainders of those divided by 7 are not 0 and distinct. For instance, $m = 2$ and $n = 1$.

Question 15. Let S denote a square of side-length 7, and let eight squares with side-length 3 be given. Show that it is impossible to cover S by those eight small squares with the condition: an arbitrary side of those (eight) squares is either coincided, parallel, or perpendicular to others of S.

Solution. For convenient, $ABCD$ is denoted by the square S. Let M, N, P, Q be the midpoints of sides of S, and O be the center of $ABCD$. Consider nine points: $A, B, C, D, M, N, P, Q, O$. Each square of the side-length 3 satisfied the condition cover at most one of those nine points. The proof is complete.