Hanoi Open Mathematical Competition 2016

Senior Section

Important:

Answer to all 15 questions.
Write your answers on the answer sheets provided.
For the multiple choice questions, stick only the letters (A, B, C, D or E) of your choice.
No calculator is allowed.

Question 1. How many are there 10-digit numbers composed from the digits 1, 2, 3 only and in which, two neighbouring digits differ by 1.

(A): 48 (B): 64 (C): 72 (D): 128 (E): None of the above.

Anwser. (B).

Question 2. Given an array of numbers $A = (672, 673, 674, \ldots, 2016)$ on table. Three arbitrary numbers $a, b, c \in A$ are step by step replaced by number $\frac{1}{3}\min(a, b, c)$. After 672 times, on the table there is only one number m, such that

(A): $0 < m < 1$ (B): $m = 1$ (C): $1 < m < 2$ (D): $m = 2$ (E): None of the above.

Anwser. (A).

Question 3. Given two positive numbers a, b such that the condition $a^3 + b^3 = a^5 + b^5$, then the greatest value of $M = a^2 + b^2 - ab$ is

(A): $\frac{1}{4}$ (B): $\frac{1}{2}$ (C): 2 (D): 1 (E): None of the above.

Anwser. (D).

Question 4. In Zoo, a monkey becomes lucky if he eats three different fruits. What is the largest number of monkeys one can make lucky having 20 oranges, 30 bananas, 40 peaches and 50 tangerines? Justify your answer.

(A): 30 (B): 35 (C): 40 (D): 45 (E): None of the above.

Anwser. (D).

Question 5. There are positive integers x, y such that $3x^2 + x = 4y^2 + y$ and $(x - y)$ is equal to

(A): 2013 (B): 2014 (C): 2015 (D): 2016 (E): None of the above.

Anwser. (E). Since $x - y$ is a square.

Solution. We have $3x^2 + x = 4y^2 + y \Leftrightarrow (x - y)(3x + 3y + 1) = y^2$.

We prove that $(x - y; 3x + 3y + 1) = 1$.

Indeed, if $d = (x - y; 3x + 3y + 1)$ then y^2 is divisible by d^2 and y is divisible by d; x is divisible by d, i.e. 1 is divisible by d, i.e. $d = 1$.

Since $x - y$ and $3x + 3y + 1$ are prime relative then $x - y$ is a perfect square.

Question 6. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that the difference of two arbitrary elements in A are different from $6, 9, 12, 15, 18, 21$. Prove that there are two elements of A for which their difference equals to 3.

Solution. Divide numbers 1, 2, ..., 106 into three groups $X = \{1, 4, 7, \ldots, 106\}$, $Y = \{2, 5, 8, \ldots, 104\}$ and $Z = \{3, 6, 9, \ldots, 105\}$. A has 16 elements, so one of the sets X, Y, Z contains at least 6 numbers from A. Without loss of generality, let X contains 6 numbers from A. Let they be $1 \leq a_1 < a_2 < \cdots < a_6 \leq 106$. Since

$$105 \geq a_6 - a_1 = (a_6 - a_5) + (a_5 - a_4) + (a_4 - a_3) + (a_3 - a_2) + (a_2 - a_1),$$

there is an index i for which $0 < a_{i+1} - a_i \leq 21$.

By the choice of X, $a_{i+1} - a_i$ is multiple of 3, so $a_{i+1} - a_i \in \{3, 6, 9, 12, 15, 18, 21\}$.

Finally, apply the given condition, it follows that $a_{i+1} - a_i = 3$, which was to be proved.

Question 7. Nine points form a grid of size 3×3. How many triangles are there with 3 vertices at these points?

Solution. We divide the triangles into two types:

Type 1: Two vertices lie in one horizontal line, the third vertex lies in another horizontal lines.

For this type we have 3 possibilities of choosing the first line, 2 possibilities of choosing the 2-nd line. In total we have $3 \times 2 \times 3 \times 3 = 54$ triangles of first type.

Type 2: Three vertices lie in distinct horizontal lines.

We have $3 \times 3 \times 3$ triangles of these type. But we should remove degenerated triangles from them. There are 5 of those (3 vertical lines and two diagonals). So, we have 27 - 5 = 22 triangles of this type.

Total, we have 54 + 22 = 76 triangles.

For those students who know about C_n^k this problem can be also solved as $C_9^3 - 8$ where 8 is the number of degenerated triangles.

Question 8. Determine all 3-digit numbers which are equal to cube of the sum of all its digits.

Solution. Let $\overline{abc}$, where $a, b, c \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $a \neq 0$ and $\overline{abc} = (a + b + c)^3$.

Note that $100 \leq (a+b+c)^3 \leq 999$ and $\sqrt[3]{100} \leq a+b+c \leq \sqrt[3]{999}$. Hence $5 \leq a+b+c \leq 9$.

If $a+b+c=5$ then $\overline{abc}=(a+b+c)^3=5^3=125$ and $a+b+c=8$ (not suitable).

If $a+b+c=6$ then $\overline{abc}=(a+b+c)^3=6^3=216$ and $a+b+c=9$ (not suitable).

If $a+b+c=7$ then $\overline{abc}=(a+b+c)^3=7^3=343$ and $a+b+c=10$ (not suitable).

If $a+b+c=8$ then $\overline{abc}=(a+b+c)^3=8^3=512$ and $a+b+c=8$ (suitable).

If $a+b+c=9$ then $\overline{abc}=(a+b+c)^3=9^3=729$ and $a+b+c=18$ (not suitable).

Conclusion: $\overline{abc}=512$.

Question 9. Let rational numbers a, b, c satisfy the conditions

$$a+b+c=a^2+b^2+c^2 \in \mathbb{Z}.$$

Prove that there exist two relative prime numbers m, n such that $abc=\dfrac{m^2}{n^3}$.

Solution. Put $a+b+c=a^2+b^2+c^2=t$.

We have $3\left(a^2+b^2+c^2\right) \geq (a+b+c)^2$, then $t \in [0;3]$.

Since $t \in \mathbb{Z}$ then $t \in \{0;1;2;3\}$.

If $t=0$ then $a=b=c=0$ and $abc=0=\dfrac{0}{1}$.

If $t=3$ then

$$(a-1)+(b-1)+(c-1)=(a-1)^2+(b-1)^2+(c-1)^2=0.$$

That follows $a=b=c=1$ and $abc=1=\dfrac{1^2}{1^3}$.

If $t=1$. Without loss of generality, assume that $c>0$;

$a=\dfrac{m_1}{n_1}; b=\dfrac{m_2}{n_2}; c=\dfrac{m_3}{n_3}; d=|n_1n_2n_3|$.

Put $\begin{cases} x=ad \\ y=bd \\ z=cd \end{cases}$ then $x;y;z \in \mathbb{Z}$ and $z>0$.

We have $\begin{cases} x+y+z=d\,(a+b+c)=d \\ x^2+y^2+z^2=d^2\left(a^2+b^2+c^2\right)=d^2 \end{cases}$

It follows $xy+yz+zx=0 \Leftrightarrow (z+x)(z+y)=z^2=c^2d^2$.

Hence, there exist $r;p;q \in \mathbb{Z}^*$ such that

$$x+z=rp^2; y+z=rq^2; z=|r|\,qp; (p;q)=1; p;q \in \mathbb{Z}^*.$$

On the other hand $d=x+y+z=r\left(p^2+q^2\right)-|r|\,pq>0$ then $r>0$.

Hence $\begin{cases} y=rq\,(q-p) \\ x=rp\,(p-q) \\ z=rpq \end{cases} \Rightarrow abc=-\dfrac{[pq(p-q)]^2}{\left(p^2+q^2-pq\right)^3}$.

We prove that $(pq(p-q); p^2+q^2-pq)=1$.

Suppose that $s=\left(pq\,(p-q)\,; p^2+q^2-pq\right); s>1$ then $s|pq\,(p-q)$.

Case 1. Let $s|p$. Since $s|\left(p^2+q^2-pq\right)$ then $s|q$ and $s=1$ (not suitable).

Case 2. Let $s|q$. Similarly, we find $s=1$ (not suitable).

Case 3. If $s|\left(p-q\right)$ then $s|(p-q)^2-(p^2+q^2-pq) \Rightarrow s|pq \Rightarrow \left[\begin{matrix} s|p \\ s|q \end{matrix}\right.$ (not suitable).

If $t=2$ then $a+b+c=a^2+b^2+c^2=2$.

We reduce it to the case where $t=1$, which was to be proved.

Question 10. Given natural numbers a, b such that $2015a^2+a=2016b^2+b$. Prove that $\sqrt{a-b}$ is a natural number.

Solution. From equality

$$2015a^2+a=2016b^2+b, \tag{1}$$

we find $a \geq b$.

If $a=b$ then from (1) we have $a=b=0$ and $\sqrt{a-b}=0$.

If $a>b$, we write (1) as

$$b^2=2015(a^2-b^2)+(a-b) \Leftrightarrow b^2=(a-b)(2015a+2015b+1). \tag{2}$$

Let $(a,b)=d$ then $a=md$; $b=nd$, where $(m,n)=1$. Since $a>b$ then $m>n$; and put $m-n=t>0$.

Let $(t,n)=u$ then n is divisible by u; t is divisible by u and m is divisible by u. That follows $u=1$ and then $(t,n)=1$.

Putting $b=nd$; $a-b=td$ in (2), we find

$$n^2d=t(2015dt+4030dn+1). \tag{3}$$

From (3) we get n^2d is divisible by t and compaire with $(t,n)=1$, it follows d is divisible by t.

Also from (3) we get $n^2d=2015dt^2+4030dnt+t$ and then $t=n^2d-2015dt^2-4030dnt$.

Hence $t=d(n^2-2015t^2-4030nt)$, i.e. t is divisible by d, i.e. $t=d$ and then $a-b=td=d^2$ and $\sqrt{a-b}=d$ is a natural number.

Question 11. Let I be the incenter of triangle ABC and ω be its circumcircle. Let the line AI intersect ω at point $D \neq A$. Let F and E be points on side BC and arc BDC respectively such that $\angle BAF=\angle CAE<\frac{1}{2}\angle BAC$. Let X be the second point of intersection of line EI with ω and T be the point of intersection of segment DX with line AF. Prove that $TF.AD=ID.AT$.

Solution.

Let the line AF intersect ω at point $K \neq A$ and L be the foot of the bisector of angle BAC. Since $\angle BAK=\angle CAE$ we have $\overset{\frown}{BK}=\overset{\frown}{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT=\angle DAK=\angle EAD=\angle IXT$, so the points I, A, X, T are concyclic. Hence, $\angle ITA=\angle IXA=\angle EXA=\angle EKA$, so $IT \parallel KE \parallel BC$.

Therefore, $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \dfrac{1}{2}\angle BAC$. Hence, the triangles DCL and DCA are similar. Therefore, $\dfrac{CL}{AC} = \dfrac{DC}{AD}$.

Finally, we have $\angle DIC = \angle IAC + \angle ICA = \angle ICL + \angle LCD = \angle ICD$. It follows DIC is a isosceles triangle at D. Hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get $\dfrac{TF}{AT} = \dfrac{IL}{AI} = \dfrac{CL}{AC} = \dfrac{DC}{AD} = \dfrac{ID}{AD} \Rightarrow \dfrac{TF}{AT} = \dfrac{ID}{AD} \Rightarrow TF.AD = ID.AT$ as desired.

Question 12. Let A be point inside the acute angle xOy. An arbitrary circle ω passes through O, A; intersecting Ox and Oy at the second intersection B and C, respectively. Let M be the midpoint of BC. Prove that M is always on a fixed line (when ω changes, but always goes through O and A).

Solution. Let $(O_x), (O_y)$ be circles passing throught O, A and tangent to Ox, Oy, respectively. Circle (O_x) intersects the ray Oy at D, distinct from O and circle (O_y) intersects the ray Ox at E, distinct from O. Let N and P be midpoint of OE and OD, respectively. Then N, P are fixed. We'll show that M, N, P are collinear. For this, it is sufficient to prove that $\dfrac{NO}{NB} = \dfrac{PO}{PC}$

Since (O_x) is tangent to $Ox, \angle ADC = \angle AOB$. Since $OBAC$ is cyclic, $\angle ABO = \angle ACD$. So triangles AOB, ADC are similar. Therefore $\dfrac{AB}{AC} = \dfrac{OB}{DC}$ (1)

Similarly, $\triangle ABE \backsim \triangle ACO$, so $\dfrac{BE}{CO} = \dfrac{AB}{AC}$ (2)

From (1) and (2), we deduce that

$$\frac{OB}{CD} = \frac{BE}{OC} \Rightarrow \frac{OB}{BE} = \frac{CD}{OC}$$

Hence

$$\frac{OE}{BE} = \frac{OD}{OC} \Rightarrow \frac{ON}{BE} = \frac{OP}{OC} \Rightarrow \frac{ON}{NB} = \frac{ON}{BE - NO} = \frac{OP}{OC - OP} = \frac{OP}{CP}$$

It follows, if NP intersects BC at M, then $\dfrac{MB}{MC} \cdot \dfrac{PC}{PO} \cdot \dfrac{NO}{NB} = 1$ (by Menelaus' Theorem in triangle OBC) conclusion $\dfrac{MB}{MC} = 1$, it follows NP passes through M is midpoint of BC.

Question 13. Find all triples (a, b, c) of real numbers such that $|2a + b| \geq 4$ and

$$|ax^2 + bx + c| \leq 1 \ \ \forall x \in [-1, 1].$$

Solution. From the assumptions, we have $|f(\pm 1)| \leq 1$, $|f(0)| \leq 1$ and

$$\begin{cases} f(1) = a + b + c \\ f(-1) = a - b + c \\ f(0) = c \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{1}{2}[f(1) + f(-1)] - f(0) \\ b = \dfrac{1}{2}[f(1) - f(-1)] \\ c = f(0) \end{cases}$$

That follows

$$4 \leq |2a+b| = \left|[f(1)+f(-1)]-2f(0)+\frac{1}{2}[f(1)-f(-1)]\right| = \left|\frac{3}{2}f(1)+\frac{1}{2}f(-1)-2f(0)\right|$$

$$\leq \frac{3}{2}|f(1)| + \frac{1}{2}|f(-1)| + 2|f(0)| \leq \frac{3}{2} + \frac{1}{2} + 2 = 4.$$

Hence $|2a + b| = 4$ and then

$$\begin{cases} |f(1)| = |a + b + c| = 1 \\ |f(-1)| = |a - b + c| = 1 \\ |f(0)| = |c| = 1 \end{cases} \Leftrightarrow \left[\begin{array}{l} (a, b, c) = (2, 0, -1) \\ (a, b, c) = (-2, 0, 1) \end{array}\right.$$

It is easely seen that both two triples $(2, 0, -1)$ and $(-2, 0, 1)$ satisfy the required conditions.

Question 14. Let $f(x) = x^2 + px + q$, where p, q are integers. Prove that there is an integer m such that

$$f(m) = f(2015).f(2016).$$

Solution. We shall prove that

$$f[f(x) + x] = f(x)f(x + 1). \qquad (1)$$

Indeed, we have

$$\begin{aligned} f[f(x) + x] &= [f(x) + x]^2 + p[f(x) + x] + q \\ &= f^2(x) + 2f(x).x + x^2 + pf(x) + px + q \\ &= f(x)[f(x) + 2x + p] + x^2 + px + q \\ &= f(x)[f(x) + 2x + p] + f(x) \\ &= f(x)[f(x) + 2x + p + 1] \\ &= f(x)[x^2 + px + q + 2x + p + 1] \\ &= f(x)[(x + 1)^2 + p(x + 1) + q] \\ &= f(x)f(x + 1), \end{aligned}$$

which proves (1).

Putting $m := f(2015) + 2015$ gives

$$f(m) = f[f(2015) + 2015] = f(2015)f(2015 + 1) = f(2015)f(2016),$$

as desired.

Question 15. Let a, b, c be real numbers satisfying the condition

$$18ab + 9ca + 29bc = 1.$$

Find the minimum value of the expression

$$T = 42a^2 + 34b^2 + 43c^2.$$

Solution. We have

$$\begin{aligned} T-&2(18ab + 9ca + 29bc) = \\ &= (5a - 3b)^2 + (4a - 3c)^2 + (4b - 5c)^2 + (a - 3b + 3c)^2 \\ &\geq 0, \ \ \forall a, b, c \in \mathbb{R}. \end{aligned}$$

That follows $T \geq 2$. The equality occures if and only if

$$\begin{cases} 5a - 3b = 0 \\ 4a - 3c = 0 \\ 4b - 5c = 0 \\ a - 3b + 3c = 0 \\ 18ab + 9ca + 29bc = 1 \end{cases} \Leftrightarrow \begin{cases} 5a - 3b = 0 \\ 4a - 3c = 0 \\ 4b - 5c = 0 \\ 18ab + 9ca + 29bc = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 3t, b = 5t, c = 4t \\ (18 \times 15 + 9 \times 12 + 29 \times 20)t^2 = 1 \end{cases}$$

$$\Leftrightarrow a = \frac{\pm 3}{\sqrt{958}}, \ \ b = \frac{\pm 5}{\sqrt{958}}, \ \ c = \frac{\pm 4}{\sqrt{958}}.$$